MA 1: PROBLEM SET NO. 1 SOLUTIONS

1. Prove that for all non-negative integers \( n, k \) with \( 0 \leq k \leq n + 1 \) one has

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.
\]

Proof. We can prove this identity directly:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n+1-k)!} + \frac{n!}{k!(n-k)!} = \frac{n!k + n!(n+1-k)}{k!(n+1-k)!} = \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.
\]

Alternate Proof. We won’t use these kinds of proofs in this class, but for simple identities between integers like this one, we can give a combinatorial proof by showing that the numbers on both sides of the equation are two different ways of counting the same quantity.

The number on the left corresponds to the number of ways of choosing \( k \) elements from \( n + 1 \) objects (without replacement). Choose a distinguished element \( A \) from these \( n + 1 \) objects. Then the first number on the right-hand side counts the number of combinations of \( k \) elements containing \( A \), and the second number counts the number of combinations of \( k \) elements that do not contain \( A \). Since these two sets are disjoint and their union gives all combinations of \( k \) objects from \( n + 1 \) objects, we deduce the equality.

If you like proofs like this, I highly recommend checking out the Math 6 series at Caltech, which is full of fun problems and emphasizes these kinds of proofs.

2. Prove that for a pair of real numbers \( a \) and \( b \), one has

\[
(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.
\]
Proof. We prove this by induction on \(n\).

**Base case.** For \(n = 1\), we have

\[
(a + b) = \left(\begin{array}{c}
1 \\
0
\end{array}\right) a^{1-0} b^0 + \left(\begin{array}{c}
1 \\
1
\end{array}\right) a^{1-1} b = a + b,
\]

so the identity holds in this case.

**Inductive step.** Assume that the identity holds for \(n\) (the induction hypothesis), we want to show that it holds for the \(n + 1\) case.

By the induction hypothesis, we have

\[
(a + b)^{n+1} = (a + b)^n (a + b) = \\
\left[ \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \right] (a + b) = \\
\sum_{i=0}^{n} \binom{n}{i} a^{n+1-i} b^i + \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^{i+1} = \\
\sum_{i=0}^{n} \binom{n}{i} a^{n+1-i} b^i + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n+1-j} b^j \quad \text{(setting } j = i + 1) = \\
a^{n+1} + \left(\sum_{i=1}^{n} \left[ \binom{n}{i} + \binom{n}{i-1} \right] a^{n+1-i} b^i \right) + b^{n+1} = \\
\left(\begin{array}{c}
1 \\
0
\end{array}\right) a^{n+1} + \left(\sum_{i=1}^{n} \binom{n+1}{i} a^{n+1-i} b^i \right) + \left(\begin{array}{c}
1 \\
1
\end{array}\right) b^{n+1} \quad \text{(by Problem 1)} = \\
\sum_{i=0}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i,
\]

so by the principle of mathematical induction, our identity holds. \(\square\)

**3.** For any pair of integers \(n, k \geq 1\), show that

\[
1^k + 2^k + \cdots + (n - 1)^k < \frac{n^{k+1}}{k+1} < 1^k + 2^k + \cdots + n^k.
\]

**Proof.** Fix a \(k \geq 1\). We will prove this result by induction on \(n\).

**Base case.** For \(n = 1\), we have

\[
(n - 1)^k = 0^k = 0 < \frac{1^{k+1}}{k+1} = \frac{1}{k+1} < 1^k = 1.
\]

**Inductive step.** Assuming that the identity holds for \(n\) (the induction hypothesis), we will prove these two inequalities for the \(n + 1\) case.
We first prove the first inequality. By using the induction hypothesis and adding $n^k$ to each side, we obtain

$$1^k + 2^k + \cdots + n^k < \frac{n^{k+1}}{k+1} + n^k = \frac{n^{k+1} + n^k(k+1)}{k+1}.$$ 

Note that

$$\text{(*)} \quad (n+1)^{k+1} = n^{k+1} + \binom{k+1}{1}n^k + \cdots + \binom{k+1}{k}n + 1.$$ 

Therefore, $n^{k+1} + n^k(k+1) < (n+1)^{k+1}$ and so

$$\frac{n^{k+1}}{k+1} + n^k < \frac{(n+1)^{k+1}}{k+1},$$

establishing the first inequality.

We now prove the second inequality. By using the induction hypothesis and adding $(n+1)^k$ to each side, we obtain

$$\frac{n^{k+1}}{k+1} + (n+1)^k = \frac{n^{k+1} + (k+1)(n+1)^k}{k+1} < 1^k + 2^k + \cdots + n^k + (n+1)^k.$$ 

Note that

$$n^{k+1} + (k+1)(n+1)^k = n^{k+1} + (k+1)\left(n^k + \binom{k}{1}n^{k-1} + \cdots + \binom{k}{k-1}n + 1\right).$$

To prove our inequality, we make the following elementary observation.

**Lemma.** For $k, m \geq 1$, we have

$$\binom{k+1}{m} < (k+1)\binom{k}{m}.$$ 

**Proof of Lemma.** We have

$$\frac{(k+1)!}{m!(k-m)!} = \frac{k!(k+1)}{m!(k-m)!} > \frac{1}{\frac{m!(k+1-m)!}{m!(k+1)!}} \diamond$$

Using the lemma and (\text{*}), we see that $n^{k+1} + (k+1)(n+1)^k > (n+1)^{k+1}$, and so

$$\frac{n^{k+1}}{k+1} + (n+1)^k > \frac{(n+1)^{k+1}}{k+1},$$

establishing the second identity.

Thus, by the principle of mathematical induction, our identity follows for all $k, n \geq 1$. \hfill \Box

4. Show by induction on $n$ that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$
Proof. We proceed by induction on $n$.

Base case. For the $n = 1$ case, we have

$$1 = \frac{1 \cdot 2 \cdot 3}{6}.$$ 

Inductive step. Assuming our identity for $n$ (the induction hypothesis), we want to deduce the identity for $n + 1$.

By using the induction hypothesis and adding $(n + 1)^2$ to each side, we obtain

$$1^2 + 2^2 + \cdots + n^2 + (n + 1)^2 = \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2$$

$$= \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6}$$

$$= \frac{(n + 1)(2n^2 + n + 6n + 6)}{6}$$

$$= \frac{(n + 1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n + 1)(n + 2)(2n + 3)}{6}$$

$$= \frac{(n + 1)(n + 2)[2(n + 1) + 1]}{6},$$

as desired. By the principle of mathematical induction, we deduce our result for all $n \geq 1$. □

5. Use the identity from the above problem to calculate the area enclosed by the $x$-axis and the graph of the function $y = x^2$ from 0 to a real number $a$. 

4
Solution.

The sum of red rectangles is given as

$$\frac{a^3}{n^3}(1^2 + 2^2 + \cdots + n^2) = \frac{n(n+1)(2n+1)a^3}{6n^3}$$

The sum of blue rectangles is given as

$$\frac{a}{n} \left[ \left( \frac{a}{n} \right)^2 + \left( \frac{2a}{n} \right)^2 + \cdots + \left( \frac{(n-1)a}{n} \right)^2 \right] = \frac{a^3}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2) = \frac{n(n-1)(2n-1)a^3}{6n^3}.$$

When $n$ goes to infinity, the right hand sides of both equations become $\frac{a^3}{3}$. Therefore the area enclosed by $x$-axis and the graph becomes $\frac{a^3}{3}$. $\square$

6. Consider the function $f(x) = \frac{1}{3}x^3$. Using the method explained in class, find the slope of the line tangent to the graph of $f$ at the point $(x, f(x))$.

Proof. We can compute the slope of the line tangent to the graph of $f$ at the point $(x, f(x))$ by calculating the slope of the line connecting $(x, f(x))$ and $(x+h, f(x+h))$ and considering $h \to 0$. 

5
The slope of the line connecting \((x, f(x))\) and \((x + h, f(x) + h)\) is equal to
\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^3 - x^3}{3h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{3h} = \frac{3x^2 + 3xh + h^2}{3}
\]
Therefore when \(h\) goes to zero, the slope of the line tangent to the graph of \(f\) at the point \((x, f(x))\) becomes \(x^2\). \(\square\)

7. (Bonus) Write a statement you would conjecture from the results of the previous two problems. Can you give an argument for your conjecture?

Solution. We can realize that the operations in Problem (6) and (7) are inverse to each other. This can be an argument supporting that integration and derivation are inverse to each other. \(\square\)

8. Let \(\epsilon > 0\) be a fixed positive number. Prove that if \(y_0 \neq 0\) and
\[
|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right),
\]
then \(y \neq 0\) and
\[
\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon.
\]
Here, given two numbers \(a, b\), \(\min(a, b)\) denotes the minimum of \(a\) and \(b\). Similarly, one uses \(\max(a, b)\) to denote the maximum of two numbers \(a\) and \(b\).

Proof. \(|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right)\) implies that
\[
\begin{align*}
(1) & \quad |y - y_0| < \frac{|y_0|}{2} \\
(2) & \quad |y - y_0| < \frac{\epsilon|y_0|^2}{2}
\end{align*}
\]
From (1), \(0 \leq ||y| - |y_0|| \leq |y - y_0| < \frac{|y_0|}{2}\). Therefore,
\[
-\frac{|y_0|}{2} < |y| - |y_0| < \frac{|y_0|}{2}
\]
This implies

\[ \frac{|y_0|}{2} < |y| \]

Now for nonzero \( y \),

\[
\left| \frac{1}{y} - 1 \right| = \frac{|y - y_0|}{|yy_0|} \quad \text{from (2)}
\]
\[
< \frac{\epsilon |y_0|^2}{2|y||y_0|} = \frac{\epsilon |y_0|}{2|y|} \quad \text{from (3)}
\]
\[
< \epsilon
\]

\[ \square \]

Alternate proof. (Brian: Here’s a proof that goes along the lines of the strategy that discussed at my office hours on Sunday.) First, we show that \( y \neq 0 \). Suppose that \( y = 0 \). Then \( |y - y_0| = |y_0| = |y_0| < \frac{|y_0|}{2} \), but this is a contradiction.

For the inequality, we first see that

\[
\left| \frac{1}{y} - 1 \right| = \left| \frac{y - y_0}{yy_0} \right| = \frac{1}{|yy_0|} |y - y_0|
\]
\[
< \frac{\epsilon |y_0|^2}{2|y||y_0|} = \frac{\epsilon |y_0|}{2|y|}.
\]

Thus, to complete our proof, it is enough to show that \( \frac{|y_0|}{|y|} \leq 2 \).

Suppose for contradiction that \( |y_0| > 2|y| \), so \( y = ay_0 \) for some \( a \in \mathbb{R} \) such that \( |a| < \frac{1}{2} \). Then

\[
|y - y_0| = |ay_0 - y_0| = |a - 1||y_0| > \frac{|y_0|}{2},
\]

which contradicts the fact that \( |y - y_0| < \frac{|y_0|}{2} \).

\[ \square \]

9. The numbers 1, 2, 3, 5, 8, 13, 21, \ldots, in which each term after the second is the sum of its two predecessors, are called Fibonacci numbers. They may be defined inductively as follows:

\[ F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2} \text{ if } n > 2. \]

Show by induction on \( n \) that

\[ F_n < \left( \frac{1 + \sqrt{5}}{2} \right)^n. \]
Proof. Base case. Since the inductive definition of Fibonacci numbers depend on two predecessors, we need to verify the claim for the first two cases as the induction basis. For $n = 1$,

$$F_1 = 1 < \frac{1 + \sqrt{5}}{2}.$$ 

Moreover, for $n = 2$,

$$F_2 = 2 < \left(\frac{1 + \sqrt{5}}{2}\right)^2 = \frac{3 + \sqrt{5}}{2}.$$ 

Inductive step. Let’s prove that when the $n - 1, n - 2$ cases are true, the $n$th case is true. Then this implies the inequality by induction. Suppose that $F_{n-1} < \left(\frac{1 + \sqrt{5}}{2}\right)^{n-1}$ and $F_{n-2} < \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2}$.

$$F_n = F_{n-1} + F_{n-2} < \left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} + \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2}$$

$$= \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{1 + \sqrt{5}}{2} + 1\right) = \left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} \left(\frac{3 + \sqrt{5}}{2}\right)$$

$$= \left(\frac{1 + \sqrt{5}}{2}\right)^n$$

$\square$