The Polynomial method: Problem Set 3

Due noon, Wednesday, June 3rd

You may use the following property in this assignment without proof: There does not exists an infinite chain

$$V_1 \supset V_2 \supset V_3 \supset \cdots$$

where the V_i's are varieties in $\mathbb{R}^d$ and the containments are all proper.

1. The following theorem is a variant of a recent paper by Hablicsek and Scherr (there is no need to look at the paper, but it can be found in `http://arxiv.org/abs/1412.7025`).

Theorem. *Let $\mathcal{L}$ be a set of lines and let $\mathcal{P}$ be a set of m points, both in $\mathbb{R}^3$, such that each line of $\mathcal{L}$ contains at least r points of $\mathcal{P}$. If $|\mathcal{L}| = \Omega\left(m^2/r^4 + m/r\right)$ then there exists a plane containing $\Omega(m/r^2)$ points of $\mathcal{P}$.*

Rely on the theorem to prove the following corollary, while also finding what the question marks should be replaced with (hint: look around Chapter 7).

Corollary. *Let $\mathcal{L}$ be a set of n lines and let $\mathcal{P}$ be a set of m points, both in $\mathbb{R}^3$, such that every plane contains $O(???)$ points of $\mathcal{P}$ and no line of $\mathcal{L}$ contains more than $m^{1/3}$ points of $\mathcal{P}$. Then $I(\mathcal{P}, \mathcal{L}) = O\left(m^{1/2}n^{3/4} + m + n\right)$.*

2. Prove Lemma 3.1 from Chapter 7 (hint: consider a generic plane). You are not allowed to use resultants in your proof (if you do not know what resultants are, ignore this sentence).

3. Prove Lemma 1.2 of Chapter 8 for the case where the lines are replaced with curves of degree d. The bound of the lemma should change from $O(n^{3/4}/\sqrt{k})$ to $O(n^{3/4}d^{3/4}/\sqrt{k})$. To save you the trouble of calculating various probabilities, you may assume that everything random behaves just like the expectation.

4. Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$, such that no four distinct points $a, b, c, d \in \mathcal{P}$ are the vertices of an isosceles trapezoid. Prove that $\mathcal{P}$ determines $\Omega(n)$ distinct distances. The following paragraph describes one way to prove this, but you do not have to rely on it.

Denote by x the number of distinct distances that are spanned by $\mathcal{P}$. Consider the set of isosceles triangles that are spanned by $\mathcal{P}$. That is, set

$$T = \{(a, b, c) \in \mathcal{P}^3 : |ab| = |ac| \text{ and } a \neq c\}.$$

Prove the claim by double counting $|T|$. Use Cauchy-Schwarz to find a lower bound for the number of triangles that involve a specific $a \in \mathcal{P}$ (this is similar to what we did in both parts of Chapter 6). For the upper bound, find a connection to perpendicular bisectors of pairs of points of $\mathcal{P}$.

Make sure that you rely on the isosceles trapezoids property, and did not accidentally prove the claim for every point set (since this would mean that your proof is wrong).