The Polynomial method: Problem Set 2

Due noon, Friday, May 15th

1. Prove that Theorem 2.1 of Chapter 3 (of the lecture notes) remains valid also after removing the restrictions about the curves being irreducible and distinct.

2. Claim 4.1 of Chapter 3 gives an upper bound for the maximum number of lattice points that an irreducible curve of degree at least two can contain. Your goal in this problem is to improve the bound of this claim to $O\left(n^{1/3}\right)$. The following paragraph instructs you how to do this, although you may use a different approach if you like.

Construct the sets $\mathcal{P}'$ and Γ as in the original proof. Then move to a different plane, as follows (this is similar to an argument that we used in the proof of Theorem 1.2 of Chapter 6). Recall that every curve $\gamma_i \in \Gamma$ is a translation of γ, which can be decomposed into a horizontal translation followed by a vertical translation. Instead of γ_i, consider a point v_i whose x-coordinate is the distance of the horizontal translation, and whose y-coordiante is the distance of the vertical one. Replace every point $p \in \mathcal{P}'$ with the set S_p of the points in $\mathbb{R}^2$ that correspond to translations of γ that are incident to p (we do not refer only to points that correspond to curves in Γ, but rather to any point that parameterizes a translated copy of γ incident to p). How does the set S_p look like?

3. Let T be a drawing of a spanning tree[1] in $\mathbb{R}^2$. The edges of T are not necessarily straight and they may not intersect each other. The *line crossing number* of T is the maximum number of edges of T the are intersected by a line that does not contain any of the vertices of T.

We claim that for every set of n distinct vertices in $\mathbb{R}^2$, it is possible to add edges, possibly together with $O(n)$ additional vertices, such that a spanning tree with a line crossing number of $O(\sqrt{n})$ is obtained (the n original vertices cannot be moved from their given positions). The following two paragraphs contain a proof for this claim. While the claim is correct, the proof has a serious mistake in it. Your goal is to find this mistake. There is no need to fix the proof (if you have an idea about how to fix it, I will be happy to hear about it!)

Proof. The vertices are n distinct points in $\mathbb{R}^2$. By Lemma 2.1 of Chapter 5, there exists a curve γ of degree $O(\sqrt{n})$ that contains all of these points. We add an edge between every two vertices that are adjacent along γ, where the edge is the segment of γ between these two vertices. The resulting graph might have cycles in it. We repeatedly consider a cycle and remove an arbitrary edge from it, until no cycles remain. By Bézout's theorem, every line intersects γ in $O(\sqrt{n})$ points, so every line intersects $O(\sqrt{n})$ of the remaining edges.

The above procedure for adding edges might result in intersecting edges, and might not be well defined in the neighborhood of points where γ intersects itself. To address this issue, before inserting the edges we add a new vertex at every problematic intersection point. For example, if two lines that are contained in γ intersect, then we add a new vertex v at the

[1] If you do not know what a spanning tree is (in graph theory) or what its basic properties are, you are welcome to ask Adam about it.

intersection point (unless such a vertex already exists). We then connect v to the vertices that are consecutive to v along the two lines. Every such intersection point is a singular point of γ. Since γ has $O(n)$ singular points, we indeed add $O(n)$ additional vertices.

4. We consider a distinct distances variant for triangles, by saying that two triangles are distinct if they are not congruent. Prove that for any set $\mathcal{P}$ of n points in the plane, the vertices of $\mathcal{P}$ span $\Omega\left(n^2\right)$ distinct triangles; by "spanning" a triangle, we mean that the three vertices of the triangle are in $\mathcal{P}$ (hint: change the ESGK framework, which may also require changing the first page of Class 7. Be careful not to waste too much time on issues that only change the bound by a constant).