The Polynomial method: Problem Set 1

Due noon, Tuesday , April 28th

This assignment is based on chapters 1–3, and you may use any material that was studied in these chapters to solve the questions in it.

1. Prove that the maximum number of right-angled triangles that can be determined by a set of n points in $\mathbb{R}^2$ is $O\left(n^{7/3}\right)$ (that is, the vertices of each of the triangles are in the point set. Hint: Look at the unit-area triangles problem at the end of Chapter 1).

2. Prove that the set $\{(x, \sin x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ is not a variety (hint: consider lines that intersect this set).

3. In class we derived the bound $O_{s,t}\left(m^{s/(2s-1)}n^{(2s-2)/(2s-1)} + m + n\right)$ for the number of incidences between m points and n distinct irreducible constant-degree curves in $\mathbb{R}^2$, when the incidence graph contains no copy of $K_{s,t}$. That is, we considered s and t as constants. In this question we consider the case where t may depend on m and n (e.g., $t = \log m$ or $t = n^\alpha$ for some $0 < \alpha < 1$). Revise the analysis from class so that it would apply to this case. There is no need to rewrite the entire proof — you may focus only the parts that need to be revised.

4. *Radon's theorem* states that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint subsets whose convex hulls intersect.[1]

We are given a set $\mathcal{P}$ of n points in $\mathbb{R}^d$ such that n is much larger than d. We say that a polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ *separates* two subsets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$ if no connected component of $\mathbb{R}^d \setminus Z(f)$ contains points from both $\mathcal{P}_1$ and $\mathcal{P}_2$ and $Z(f) \cap \mathcal{P} = \emptyset$. Use Radon's theorem to prove that $\mathcal{P}$ can be partitioned into two disjoint subsets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$ such that all of the polynomials that separate $\mathcal{P}_1$ and $\mathcal{P}_2$ are of degree $\Omega_d(n^{1/d})$ (hint: two convex hulls of point sets are disjoint if and only if no hyperplane separates them. You may rely on this property without proving it).

5. Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^d$. We use a polynomial partitioning to obtain $O(r^d)$ cells, each containing at most m/r^d points of $\mathcal{P}$ (for some $1 < r < m$). Unfortunately, for the application that we have in mind, we also require the property that no cell is intersected by many lines of $\mathcal{L}$.

Show that we can further partition the existing $O(r^d)$ cells so that (i) every new cell is intersected by at most n/r^{d-1} lines of $\mathcal{L}$, and (ii) the number of cells remains $O(r^d)$. Do this by partitioning each cell C into several "abstract" subcells (i.e., different subcells do not necessarily correspond to different geometric areas), where each subcell of C consists of the same set of points as C but only of a subset of the lines. As before, every point-line incidence is required to appear in exactly one subcell (unless it is on the original partitioning, in which case it is in none of the cells and subcells).

[1] If you are not familiar with the definition of a convex hull and the Wikipedia definition is not completely clear to you, feel free to ask Adam.