Chapter 5: The Joints Problem

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In some sense the topic of this course originated from a paper by Guth and Katz [3], in which they used polynomial methods to solve two problems. One of these problems is the *joints problem*, and this short chapter is dedicated to it.

1 Introducing the problem

Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$. A *joint* of $\mathcal{L}$ is a point of $\mathbb{R}^3$ that is incident to three lines $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$ (and possibly to additional lines of $\mathcal{L}$) such that no plane fully contains ℓ_1, ℓ_2, and ℓ_3 (equivalently, such that the directions of ℓ_1, ℓ_2, and ℓ_3 are linearly independent). The joints problem asks for the maximum number of joints in a set of n lines.

Consider $n/3$ lines in the direction of the x-axis, $n/3$ lines in the direction of the y-axis, and $n/3$ lines in the direction of the z-axis, and notice that we can place these lines so that all of the points of

$$\left\{(a,b,c) \in \mathbb{N}^3 : 1 \le a,b,c \le \sqrt{n/3}\right\}$$

are joints. That is, there are sets of n lines that yield $\Theta(n^{3/2})$ joints.

The joints problem seems to have started as a discrete geometry problem (e.g., see [1]), but over the years it also attracted the attention of researchers from the analysis community. Wolff [6] observed a connection between the joints problem and the Kakeya problem. After a sequence of increasingly better bounds, the problem was completely solved by Guth and Katz.

Theorem 1.1 (Guth and Katz [3]). *The maximum number of joints in a set of n lines in $\mathbb{R}^3$ is $\Theta(n^{3/2})$.*

A generalization of the joints problem to $\mathbb{R}^d$ was independently derived by Kaplan, Sharir, and Shustin [4] and by Quilodrán [5].

2 Proving Theorem 1.1

Notice that we only need to prove the upper bound of Theorem 1.1. For this, we follow a simple proof by Guth [2].

We begin by deriving two useful lemmas. In previous chapters we were interested in polynomials that partition a point set into "well-behaved" cells. We now move to consider polynomials that fully contain a point set.

Lemma 2.1. *Given a set $\mathcal{P}$ of m points in $\mathbb{R}^d$ and a positive integer D such that $\binom{d+D}{d} > m$, there exists a nontrivial polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree D such that $\mathcal{P} \subset Z(f)$.*

Proof. In the proof of the polynomial ham sandwich theorem (Theorem 3.2 of Chapter 3), we proved that a polynomial in $\mathbb{R}[x_1, \ldots, x_d]$ of degree D can have at most $\binom{D+d}{d}$ distinct monomials. Set $k = \binom{D+d}{d}$ and recall that $k > m$. Consider a polynomial f of degree D, and denote the coefficients of the monomials of f as $c_1, \ldots, c_k$.

Asking for f to vanish on a point $p \in \mathcal{P}$ corresponds to a linear homogeneous equation in $c_1, \ldots, c_k$. Thus, we have a set of m linear homogenous equations in k variables. Since $k > m$, the system must have a non-trivial solution. Such a solution corresponds to a choice of coefficients for f such that f vanishes on $\mathcal{P}$. $\square$

Lemma 2.2. *Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ and let $\mathcal{J}$ be the set of joints of $\mathcal{L}$. Then there exists a line of $\mathcal{L}$ that contains at most $3|\mathcal{J}|^{1/3}$ of the joints.*

Proof. Assume, for contradiction, that every line of $\mathcal{L}$ contains more than $3|\mathcal{J}|^{1/3}$ joints. Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a minimum degree non-trivial polynomial that vanishes on all the points of $\mathcal{J}$. By Lemma 2.1, we have $\deg f \le 3|\mathcal{J}|^{1/3}$.

Consider a line $\ell \in \mathcal{L}$ and a generic plane h that fully contains ℓ. Notice that $\gamma = Z(f) \cap h$ is a variety of dimension at most one and of degree at most $\deg f \le 3m^{1/3}$. By Bézout's theorem (Theorem 5.1 of Chapter 2), either γ fully contains ℓ or $|\gamma \cap \ell| \le 3|\mathcal{J}|^{1/3}$. By assumption ℓ contains more than $3|\mathcal{J}|^{1/3}$ points, so we must have $\ell \subseteq \gamma$. That is, $Z(f)$ fully contains every line of $\mathcal{L}$.

Consider a point $p \in \mathcal{J}$, and let $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$ be three lines that are incident to p and are not fully contained in a common plane. By the above, we have $\ell_1, \ell_2, \ell_3 \subset Z(f)$. If p is a regular point of $Z(f)$, then the tangent plane to $Z(f)$ at p must fully contain ℓ_1, ℓ_2, ℓ_3, which is a contradiction. Thus, p is a singular point of f, which in turn implies that $\nabla f(p) = 0$ for every $p \in \mathcal{J}$.

Since f is nontrivial, at least one of its first partial derivatives is not identically zero. Without loss of generality, we assume that this is the case for $f_1 = \frac{\partial f}{\partial x_1}$. By the above property of ∇f, we have that f_1 vanishes on every point of $\mathcal{J}$. This a contradiction to f being a minimum degree polynomial that vanishes on $\mathcal{P}$, which completes the proof of the lemma. $\square$

After deriving Lemma 2.2, it is straightforward to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\mathcal{L}$ be a set of n lines in $\mathbb{R}^3$, let $\mathcal{J}$ be the set of joints of $\mathcal{L}$, and put $x = |\mathcal{J}|$. We repeatedly consider a line that is incident to at most $3x^{1/3}$ joints of $\mathcal{J}$, remove this line from $\mathcal{L}$, and update $\mathcal{J}$ accordingly (the value of x remains fixed during this process). By Lemma 2.2, such a line exists at every step. Since every line removal destroys at most $3x^{1/3}$ joints, and since after removing all of the lines no joint remains, we have

$$x \le n \cdot 3x^{1/3}.$$

The assertion of the theorem is obtained by tidying up this equation. □

It is straightforward to extend the proof of Theorem 1.1 to joints in $\mathbb{R}^d$.

References

[1] B. Chazelle, H. Edelsbrunner, L. J. Guibas, R. Pollack, R. Seidel, M. Sharir, and J. Snoeyink, Counting and cutting cycles of lines and rods in space, *Computational Geometry* 1 (1992), 305–323.

[2] L. Guth, Lecture notes for the course "The Polynomial Method", `http://math.mit.edu/~lguth/PolynomialMethod.html`.

[3] L. Guth, and N. H. Katz, Algebraic methods in discrete analogs of the Kakeya problem, Advances in Mathematics **225.5** (2010), 2828–2839.

[4] H. Kaplan, M. Sharir, and E. Shustin, On lines and joints, *Discrete Comput. Geom.* **44** (2010), 838–843.

[5] R. Quilodrán, The Joints Problem in $\mathbb{R}^n$, *SIAM Journal on Discrete Math.* **23** (2010), 2211–2213.

[6] T. Wolff, Recent work connected with the Kakeya problem, *Prospects in mathematics* (Princeton, NJ, 1996) 2 (1999), 129–162.