Chapter 5: The Joints Problem

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In some sense the topic of this course originated from a paper by Guth and Katz [3], in which they used polynomial methods to solve two problems. One of these problems is the joints problem, and this short chapter is dedicated to it.

1 Introducing the problem

Let $L$ be a set of lines in $\mathbb{R}^3$. A joint of $L$ is a point of $\mathbb{R}^3$ that is incident to three lines $\ell_1, \ell_2, \ell_3 \in L$ (and possibly to additional lines of $L$) such that no plane fully contains $\ell_1, \ell_2,$ and $\ell_3$ (equivalently, such that the directions of $\ell_1, \ell_2,$ and $\ell_3$ are linearly independent). The joints problem asks for the maximum number of joints in a set of $n$ lines.

Consider $n/3$ lines in the direction of the $x$-axis, $n/3$ lines in the direction of the $y$-axis, and $n/3$ lines in the direction of the $z$-axis, and notice that we can place these lines so that all of the points of

$$\left\{(a, b, c) \in \mathbb{N}^3 : 1 \leq a, b, c \leq \sqrt{n/3}\right\}$$

are joints. That is, there are sets of $n$ lines that yield $\Theta(n^{3/2})$ joints.

The joints problem seems to have started as a discrete geometry problem (e.g., see [1]), but over the years it also attracted the attention of researchers from the analysis community. Wolff [6] observed a connection between the joints problem and the Kakeya problem. After a sequence of increasingly better bounds, the problem was completely solved by Guth and Katz.

Theorem 1.1 (Guth and Katz [3]). The maximum number of joints in a set of $n$ lines in $\mathbb{R}^3$ is $\Theta(n^{3/2})$.

A generalization of the joints problem to $\mathbb{R}^d$ was independently derived by Kaplan, Sharir, and Shustin [4] and by Quilodrán [5].

2 Proving Theorem 1.1

Notice that we only need to prove the upper bound of Theorem 1.1. For this, we follow a simple proof by Guth [2].
We begin by deriving two useful lemmas. In previous chapters we were interested in polynomials that partition a point set into “well-behaved” cells. We now move to consider polynomials that fully contain a point set.

**Lemma 2.1.** Given a set $\mathcal{P}$ of $m$ points in $\mathbb{R}^d$ and a positive integer $D$ such that $\binom{D+d}{d} > m$, there exists a nontrivial polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $D$ such that $\mathcal{P} \subset Z(f)$.

**Proof.** In the proof of the polynomial ham sandwich theorem (Theorem 3.2 of Chapter 3), we proved that a polynomial in $\mathbb{R}[x_1, \ldots, x_d]$ of degree $D$ can have at most $\binom{D+d}{d}$ distinct monomials. Set $k = \binom{D+d}{d}$ and recall that $k > m$. Consider a polynomial $f$ of degree $D$, and denote the coefficients of the monomials of $f$ as $c_1, \ldots, c_k$.

Asking for $f$ to vanish on a point $p \in \mathcal{P}$ corresponds to a linear homogeneous equation in $c_1, \ldots, c_k$. Thus, we have a set of $m$ linear homogeneous equations in $k$ variables. Since $k > m$, the system must have a non-trivial solution. Such a solution corresponds to a choice of coefficients for $f$ such that $f$ vanishes on $\mathcal{P}$. □

**Lemma 2.2.** Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ and let $\mathcal{J}$ be the set of joints of $\mathcal{L}$. Then there exists a line of $\mathcal{L}$ that contains at most $3|\mathcal{J}|^{1/3}$ of the joints.

**Proof.** Assume, for contradiction, that every line of $\mathcal{L}$ contains more than $3|\mathcal{J}|^{1/3}$ joints. Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a minimum degree non-trivial polynomial that vanishes on all the points of $\mathcal{J}$. By Lemma 2.1, we have $\deg f \leq 3|\mathcal{J}|^{1/3}$.

Consider a line $\ell \in \mathcal{L}$ and a generic plane $h$ that fully contains $\ell$. Notice that $\gamma = Z(f) \cap h$ is a variety of dimension at most one and of degree at most $\deg f \leq 3m^{1/3}$. By Bézout’s theorem (Theorem 5.1 of Chapter 2), either $\gamma$ fully contains $\ell$ or $|\gamma \cap \ell| \leq 3|\mathcal{J}|^{1/3}$. By assumption $\ell$ contains more than $3|\mathcal{J}|^{1/3}$ points, so we must have $\ell \subset \gamma$. That is, $Z(f)$ fully contains every line of $\mathcal{L}$.

Consider a point $p \in \mathcal{J}$, and let $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$ be three lines that are incident to $p$ and are not fully contained in a common plane. By the above, we have $\ell_1, \ell_2, \ell_3 \subset Z(f)$. If $p$ is a regular point of $Z(f)$, then the tangent plane to $Z(f)$ at $p$ must fully contain $\ell_1, \ell_2, \ell_3$, which is a contradiction. Thus, $p$ is a singular point of $f$, which in turn implies that $\nabla f(p) = 0$ for every $p \in \mathcal{J}$.

Since $f$ is nontrivial, at least one of its first partial derivatives is not identically zero. Without loss of generality, we assume that this is the case for $f_1 = \frac{\partial f}{\partial x_1}$. By the above property of $\nabla f$, we have that $f_1$ vanishes on every point of $\mathcal{J}$. This a contradiction to $f$ being a minimum degree polynomial that vanishes on $\mathcal{P}$, which completes the proof of the lemma. □

After deriving Lemma 2.2, it is straightforward to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $\mathcal{L}$ be a set of $n$ lines in $\mathbb{R}^3$, let $\mathcal{J}$ be the set of joints of $\mathcal{L}$, and put $x = |\mathcal{J}|$. We repeatedly consider a line that is incident to at most $3x^{1/3}$ joints of $\mathcal{J}$, remove this line from $\mathcal{L}$, and update $\mathcal{J}$ accordingly (the value of $x$ remains fixed during this process). By Lemma 2.2, such a line exists at every step. Since every line removal destroys at most $3x^{1/3}$ joints, and since after removing all of the lines no joint remains, we have

$$x \leq n \cdot 3x^{1/3}.$$
The assertion of the theorem is obtained by tidying up this equation.

It is straightforward to extend the proof of Theorem 1.1 to joints in $\mathbb{R}^d$.

References


