Chapter 4: Constant-degree Polynomial Partitioning

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In this chapter we present a different way of deriving incidence bounds by using polynomial partitioning. This method yields slightly worse bounds but makes it significantly simpler to derive incidence bounds in $\mathbb{R}^d$ for $d \geq 3$.

1 Motivation: Incidences in higher dimensions

To see the issues that arise when studying incidence problems in higher dimensions, we consider one of the simplest cases: Incidences between m points and n planes in $\mathbb{R}^3$. To see that this problem is not interesting, we consider the following point-plane configuration. Let $\ell \subset \mathbb{R}^3$ be a line, let $\mathcal{P}$ be a set of m points on ℓ, and let $\mathcal{H}$ be a set of n planes that contain ℓ (e.g., see Figure 1). This construction satisfies $I(\mathcal{P}, \mathcal{H}) = mn$, implying that the problem is trivial.

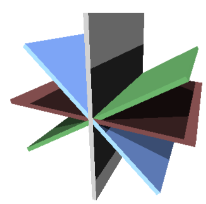

Figure 1: By having all the planes contain a given line, we can obtain mn point-plane incidences.

There are several ways to turn the problem into a non-trivial one with various applications. We consider the problem that is obtained by adding the restriction that the incidence graph of $\mathcal{P} \times \mathcal{H}$ does not contain a copy of $K_{s,t}$. This problem is interesting (and open), and to obtain some bound for it we try to adapt our proof for the case of point-curve incidences in $\mathbb{R}^2$.

By inspecting our weak incidence bound (Lemma 2.3 of Chapter 3), we notice that it is not only valid for planar curves, but also for any set of varieties in $\mathbb{R}^d$. In fact, this lemma is just a bound on the number of edges in a bipartite graph with no copy of $K_{s,t}$, and has nothing geometric about it. We thus have the weak bound $I(\mathcal{P}, \mathcal{H}) = O_{s,t}\left(mn^{1-\frac{1}{s}} + n\right)$. The polynomial partitioning theorem also applies in any dimension, and we can use it to partition $\mathcal{P}$. In fact, repeating the analysis for the incidences in the cells is straightforward. However, handling incidences on the partitioning itself becomes rather difficult. The partition is a two-dimensional variety of a large degree, and it can contain many of the points and intersect many of the planes. It is not easy to bound the number of incidences in this case, and it becomes increasingly difficult in higher dimensions.

In this chapter we discuss a method for handling incidences in higher dimensions, which was introduced by Solymosi and Tao [4]. The basic idea in this method is to use a partitioning polynomial of a constant degree (that is, the degree does not depend on the size of the input). When dealing with such a partition, handling the number of incidences on the partition becomes much simpler. However, currently it is not known how to apply this technique without losing an ε in the exponent of the incidence bound.

2 The Szemerédi-Trotter theorem yet again

Instead of immediately considering incidences in higher dimensions, we first use *constant-degree partitioning polynomials*[1] to prove a weaker version of the Szemerédi-Trotter theorem. This allows us to see how the technique works without also handling the additional issues that arise in higher dimensions.

Theorem 2.1. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then for any $\varepsilon > 0$, we have $I(\mathcal{P},\mathcal{L}) = O_\varepsilon(m^{2/3+\varepsilon}n^{2/3} + m + n)$.*

Proof. We prove the theorem by induction on $m+n$. Specifically, we prove by induction that, for any fixed $\varepsilon > 0$, there exist constants α_1, α_2 such that

$$I(\mathcal{P},\mathcal{L}) \le \alpha_1 m^{2/3+\varepsilon}n^{2/3} + \alpha_2(m+n).$$

For the induction basis, the bound holds for small $m+n$ (e.g., for $m+n \le 100$) by taking α_1 and α_2 to be sufficiently large.

For the induction step, we first recall our weak incidence bound (Lemma 2.3 of chapter 3), which implies $I(\mathcal{P},\mathcal{L}) = O(m\sqrt{n} + n)$. This completes the proof of the theorem if $m = O(\sqrt{n})$ (the resulting bound is $O(n)$ in this case). Thus, we may assume that

$$n = O\left(m^2\right). \tag{1}$$

We take r to be a sufficiently large constant, whose value depends on ε and will be determined below. Let f be an r-partitioning polynomial of $\mathcal{P}$. According to the polynomial partitioning theorem, f is of degree $O(r)$ and $Z(f)$ partitions $\mathbb{R}^2$ into connected cells, each containing at most m/r^2 points of $\mathcal{P}$. By Warren's theorem (Theorem 1.2 of Chapter 3), the number of cells is $c = O(r^2)$. The relations between

[1] As usual, there is no standard name for this technique. Some papers refer to it as "low degree polynomial partitioning".

the constants of this proof are[2]

$$2^{\varepsilon} \ll r \ll \alpha_2 \ll \alpha_1.$$

Let $\mathcal{L}_0$ denote the subset of lines of $\mathcal{L}$ that are fully contained in $Z(f)$, and let $\mathcal{P}_0 = \mathcal{P} \cap Z(f)$. Denote the cells of the partition as $K_1, \ldots, K_c$. For $i = 1, \ldots, c$, put $\mathcal{P}_i = \mathcal{P} \cap K_i$ and let $\mathcal{L}_i$ denote the set of lines of $\mathcal{L}$ that intersect K_i. Notice that

$$I(\mathcal{P}, \mathcal{L}) = I(\mathcal{P}_0, \mathcal{L}_0) + I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) + \sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i). \tag{2}$$

For any line $\ell \in \mathcal{L} \setminus \mathcal{L}_0$, by Bézout's theorem (Theorem 5.1 of Chapter 2), ℓ and $Z(f)$ have $O(r)$ common points. This immediately implies

$$I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) = O(nr). \tag{3}$$

Set $m_0 = |\mathcal{P}_0|$ and $m' = m - m_0$; that is, m' is the number of points of $\mathcal{P}$ that are in the cells. Since f is of degree $O(r)$, we get that $Z(f)$ can fully contain at most $O(r)$ lines. This in turn implies

$$I(\mathcal{P}_0, \mathcal{L}_0) = O(m_0 r). \tag{4}$$

It remains to bound $\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i)$. For $i = 1, \ldots, c$, put $m_i = |\mathcal{P}_i|$ and $n_i = |\mathcal{L}_i|$. Note that $m' = \sum_{i=1}^{c} m_i$, and recall that $m_i \le m/r^2$ for every $1 \le i \le c$. By the induction hypothesis, we have

$$\begin{aligned} \sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) &\le \sum_{i=1}^{c} \left(\alpha_1 m_i^{2/3+\varepsilon} n_i^{2/3} + \alpha_2 (m_i + n_i) \right) \\ &\le \alpha_1 \left(\frac{m}{r^2} \right)^{2/3+\varepsilon} \sum_{i=1}^{c} n_i^{2/3} + \alpha_2 \left(m' + \sum_{i=1}^{c} n_i \right). \end{aligned} \tag{5}$$

The above bound of $O(r)$ on the number of intersection points between a line $\ell \in \mathcal{L} \setminus \mathcal{L}_0$ and $Z(f)$ implies that each line enters $O(r)$ cells (a line has to intersect $Z(f)$ when moving from one cell to another). This implies $\sum_{i=1}^{c} n_i = O(nr)$. Combining this with Hölder's inequality (e.g., see Chapter 3 just before Lemma 2.3) implies

$$\sum_{i=1}^{c} n_i^{2/3} = O\left((nr)^{2/3} \cdot r^{2/3} \right) = O\left(n^{2/3} r^{4/3} \right). \tag{6}$$

By combining (5) and (6), we obtain

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) = O\left(\frac{\alpha_1 m^{2/3+\varepsilon} n^{2/3}}{r^{2\varepsilon}} + \alpha_2 nr \right) + \alpha_2 m'.$$

[2]The expression $a \ll b$ means that we take b to be sufficiently larger than a, so that some corresponding inequalities would hold.

Combining this with (3) and (4) yields

$$I(\mathcal{P},\mathcal{L}) = O\left(\frac{\alpha_1 m^{2/3+\varepsilon}n^{2/3}}{r^{2\varepsilon}} + \alpha_2 nr + m_0 r\right) + \alpha_2 m'.$$

By taking α_2 to be sufficiently large with respect to r and the constant in the $O(\cdot)$-notation, we get

$$\begin{aligned} I(\mathcal{P},\mathcal{L}) &= O\left(\frac{\alpha_1 m^{2/3+\varepsilon}n^{2/3}}{r^{2\varepsilon}} + \alpha_2 nr\right) + \alpha_2(m' + m_0) \\ &= O\left(\frac{\alpha_1 m^{2/3+\varepsilon}n^{2/3}}{r^{2\varepsilon}} + \alpha_2 nr\right) + \alpha_2 m. \end{aligned} \tag{7}$$

Recall that (1) states that $n = O(m^2)$. This implies $n = n^{2/3}n^{1/3} = O(m^{2/3}n^{2/3})$. By taking α_1 to be sufficiently large with respect to α_2, r, and the constant in the $O(\cdot)$-notation in (7), we obtain $O(\alpha_2 nr) \le \frac{\alpha_1}{2} m^{2/3}n^{2/3}$. Similarly, by taking r to be sufficiently large with respect to ε and the constant in the $O(\cdot)$-notation in (7), we may assume that

$$O\left(\frac{\alpha_1 m^{2/3+\varepsilon}n^{2/3}}{r^{2\varepsilon}}\right) \le \frac{\alpha_1}{2} m^{2/3+\varepsilon}n^{2/3}.$$

Combining this with (7) completes the induction step, and thus the proof of the theorem. □

Remarks. (i) Already in $\mathbb{R}^2$ it is simpler to handle incidences on the partition when it is of a constant degree.

(ii) Without the extra ε in the exponent of the bound of Theorem 2.1, the induction step would have failed. Specifically, when using the induction hypothesis to sum up the incidences inside of the cells, we would have obtained an expression that has the correct asymptotic value, but with a leading constant that is larger than the one we started with. A similar situation always occurs when using the constant-degree partitioning technique, and this seems to be the main disadvantage of this technique.

(iii) We still rely on the weak combinatorial bound (Lemma 2.3 of Chapter 3). Although we do not apply this bound in every cell as before, it is required in a different part of the proof.

3 The Szemerédi-Trotter theorem in $\mathbb{C}^2$

Now that we have some understanding of constant-degree partitioning polynomials, we use them to handle a more difficult problem — the Szemerédi-Trotter theorem in $\mathbb{C}^2$. That is, we have a set $\mathcal{P} \subset \mathbb{C}^2$ of m points and a set $\mathcal{L}$ of n lines in $\mathbb{C}^2$. One can think of a complex line as the zero set (over the complex numbers) of a linear polynomial with coefficients in $\mathbb{C}$.

Theorem 3.1. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{C}^2$. Then for any $\varepsilon > 0$, we have $I(\mathcal{P},\mathcal{L}) = O_\varepsilon(m^{2/3+\varepsilon}n^{2/3} + m + n)$.*

To prove Theorem 3.1, we consider $\mathbb{C}^2$ as $\mathbb{R}^4$. That is, a point $(a+ib, c+id) \in \mathbb{C}^2$ is considered as the point $(a, b, c, d) \in \mathbb{R}^4$, and the set $\mathcal{P}$ becomes a set of m points in $\mathbb{R}^4$. Given a line ℓ in $\mathbb{C}^2$, we can write $\ell = Z((a+ia')y + (b+ib')x + (c+ic'))$ for some constants $a, a', b, b', c, c' \in \mathbb{R}$. A point $(x_1 + ix_2, x_3 + ix_4) \in \mathbb{C}^2$ is in ℓ if and only if

$$(a+ia')(x_3+ix_4) + (b+ib')(x_1+ix_2) + (c+ic') = 0,$$

or equivalently,

$$bx_1 - b'x_2 + ax_3 - a'x_4 + c = 0 \quad \text{and} \quad b'x_1 + bx_2 + a'x_3 + ax_4 + c' = 0.$$

Thus, when considering ℓ as being in $\mathbb{R}^4$, it is defined by two linear equations. It is not difficult to verify that each equation defines a distinct hyperplane, and that these two hyperplanes are not parallel. That is, ℓ becomes a 2-flat in $\mathbb{R}^4$.

We reduced the complex Szemerédi-Trotter problem to an incidence problem between a set $\mathcal{P}$ of m points and a set $\mathcal{H}$ of n 2-flats, both in $\mathbb{R}^4$. If these were general 2-flats, we could have obtained mn incidences by using the same construction as in Section 1. Fortunately, 2-flats that arise from complex lines have additional properties, and we rely on the following one. Since two complex lines in $\mathbb{C}^2$ intersect in at most one point, any two 2-flats of $\mathcal{H}$ intersect in at most one point. We are now ready to use constant-degree polynomial partitioning.

Theorem 3.2. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{H}$ be a set of n 2-flats, both in $\mathbb{R}^2$, such that any two 2-flats of $\mathcal{H}$ intersect in at most one point. Then for any $\varepsilon > 0$ we have $I(\mathcal{P}, \mathcal{H}) = O_\varepsilon(m^{2/3+\varepsilon}n^{2/3} + m + n)$.*

Notice that Theorem 3.1 is an immediate corollary of Theorem 3.2.

Proof. We imitate the proof of Theorem 2.1, although this requires handling several new issues that arise in $\mathbb{R}^4$. We prove the theorem induction on $m+n$. Specifically, we prove by induction that, for any fixed $\varepsilon > 0$, there exist constants α_1, α_2 such that

$$I(\mathcal{P}, \mathcal{H}) \le \alpha_1 m^{2/3+\varepsilon}n^{2/3} + \alpha_2(m+n).$$

For the induction basis, the bound holds for small $m+n$ (e.g., for $m+n \le 100$) by choosing α_1 and α_2 sufficiently large.

For the induction step, we notice that the restriction on the 2-flats of $\mathcal{H}$ implies that the incidence graph contains no copy of $K_{2,2}$. As before, our weak incidence bound (Lemma 2.3 of chapter 3) implies $I(\mathcal{P}, \mathcal{H}) = O(m\sqrt{n} + n)$. This completes the proof of the theorem if $m = O\left(\sqrt{n}\right)$ (the resulting bound is $O(n)$ in this case). Thus we may assume that

$$n = O\left(m^2\right). \tag{8}$$

We take r to be a sufficiently large constant, whose value depends on ε and will be determined below. Let f be an r-partitioning polynomial of $\mathcal{P}$. According to the polynomial partitioning theorem, f is of degree $O(r)$ and $Z(f)$ partitions $\mathbb{R}^4$ into connected cells, each containing at most m/r^4 points of $\mathcal{P}$. By Warren's theorem

(Theorem 1.2 of Chapter 3), the number of cells is $c = O(r^4)$. As in the proof of Theorem 2.1, the relations between the constants of this proof are

$$2^{\varepsilon} \ll r \ll \alpha_2 \ll \alpha_1.$$

Denote the cells of the partition as $K_1, \ldots, K_c$. For $i = 1, \ldots, c$, put $\mathcal{P}_i = \mathcal{P} \cap K_i$ and let $\mathcal{H}_i$ denote the set of 2-flats of $\mathcal{H}$ that intersect K_i. Let $\mathcal{H}_0$ denote the subset of 2-flats of $\mathcal{H}$ that are fully contained in $Z(f)$, and let $\mathcal{P}_0 = \mathcal{P} \cap Z(f)$. Notice that

$$I(\mathcal{P}, \mathcal{H}) = I(\mathcal{P}_0, \mathcal{H}_0) + I(\mathcal{P}_0, \mathcal{H} \setminus \mathcal{H}_0) + \sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i). \tag{9}$$

Unlike in the planar case, we cannot rely on Bézout's theorem (Theorem 5.1 of Chapter 2) to bound the number of cells that are intersected by a 2-flat $h \in \mathcal{H}$. Instead, we use the following theorem.

Theorem 3.3 (Barone and Basu [1]). *Let U, W be varieties in $\mathbb{R}^d$ such that $\dim U = d'$, $\deg U = k_U$, W is defined by a single polynomial of degree k_W, and $k_W \geq 2k_U$. Then the number of connected components of $U \setminus W$ is $O_d\left(k_W^{d'} k_U^{d-d'}\right)$.*

Notice that every cell of the partition that is intersected by a 2-flat $h \in \mathcal{H}$ corresponds to at least one connected component of $h \setminus Z(f)$. By Theorem 3.3 with $U = h$ and $W = Z(f)$, we get that h intersects $O(r^2)$ cells.

Bounding $\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i)$. For $i = 1, \ldots, c$, put $m_i = |\mathcal{P}_i|$ and $n_i = |\mathcal{H}_i|$. We also set $m' = \sum_{i=1}^{c} m_i$, and recall that $m_i \leq m/r^4$ for every $1 \leq i \leq c$. By the induction hypothesis, we have

$$\begin{aligned}
\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i) &\leq \sum_{i=1}^{c} \left(\alpha_1 m_i^{2/3+\varepsilon} n_i^{2/3} + \alpha_2(m_i + n_i)\right) \\
&\leq \alpha_1 \left(\frac{m}{r^4}\right)^{2/3+\varepsilon} \sum_{i=1}^{c} n_i^{2/3} + \alpha_2 \left(m' + \sum_{i=1}^{c} n_i\right).
\end{aligned} \tag{10}$$

The above bound of $O(r^2)$ on the number of cells that are intersected by a 2-flat implies $\sum_{i=1}^{c} n_i = O(nr^2)$. Combining this with Hölder's inequality (e.g., see Chapter 3 just before Lemma 2.3) implies

$$\sum_{i=1}^{c} n_i^{2/3} = O\left(\left(nr^2\right)^{2/3} \cdot r^{4/3}\right) = O\left(n^{2/3} r^{8/3}\right). \tag{11}$$

By combining (10) and (11), we obtain

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i) = O\left(\frac{\alpha_1 m^{2/3+\varepsilon} n^{2/3}}{r^{4\varepsilon}} + \alpha_2 n r^2\right) + \alpha_2 m'.$$

As in the proof of Theorem 2.1, Equation (8) implies $n = O(m^{2/3}n^{2/3})$. Thus, by taking α_1 to be sufficiently large with respect to α_2 and r, we have

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i) = O\left(\frac{\alpha_1 m^{2/3+\varepsilon}n^{2/3}}{r^{4\varepsilon}}\right) + \alpha_2 m'.$$

Finally, by taking r to be sufficiently large with respect to ε and the constant of the $O(\cdot)$-notation, we have

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{H}_i) \le \frac{\alpha_1}{3} m^{2/3+\varepsilon}n^{2/3} + \alpha_2 m'. \tag{12}$$

Bounding $I(\mathcal{P}_0, \mathcal{H} \setminus \mathcal{H}_0)$. If a 2-flat $h \in \mathcal{H}$ is not fully contained in $Z(f)$, then $Z(f) \cap h$ is at most one-dimensional. Specifically, $Z(f) \cap h$ is a variety of dimension at most one and of degree $O(r)$ (since it is defined by the linear equations that define h and by f). We denote the set of these lower-dimensional varieties as $\Gamma = \{h \cap Z(f) : h \in \mathcal{H} \setminus \mathcal{H}_0\}$. We perform a generic rotation of $\mathbb{R}^4$ around the origin and then project $\mathcal{P}_0$ and Γ onto the x_1x_2-plane. That is, we apply the projection $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2)$ (a generic rotation followed by the projection π is equivalent to a projection onto a generic 2-flat of $\mathbb{R}^4$).

We sidetrack from the proof for a quick discussion about projections of varieties. Let $U \subset \mathbb{R}^d$ be a variety of degree k and of dimension d', and let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be a projection. The image of the projection $\pi(U)$ is not necessarily a variety. For example, the image of the projection of $V(xy-1) \subset \mathbb{R}^2$ onto the x-axis is the set $\{x \in \mathbb{R}^2 : x \neq 0\}$, which is not a variety (see Figure 2). However, $\pi(U)$ is contained in a variety of dimension at most d' (e.g., see [2, Proposition 2.8.6]) and of degree $O_{k,d}(1)$.[3] We denote this variety as $\overline{\pi(U)}$.

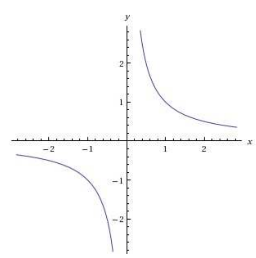

Figure 2: The curve $\mathbf{V}(xy-1)$.

Returning to the proof of Theorem 3.2, we set $|\mathcal{P}_0| = m_0$,

$$\mathcal{P}' = \{\pi(p) : p \in \mathcal{P}\}, \quad \text{and} \quad \Gamma' = \{\overline{\pi(\gamma)} : \gamma \in \Gamma\}.$$

The projection of Γ might result in new intersection points between the varieties of Γ'. However, since we first perform a generic rotation, we may assume that the projection does not lead to new incidences, and that the points of $\mathcal{P}'$ and curves of

[3] I still need to add a proper reference for the degree property.

Γ' are distinct. That is, the incidence graph of $\mathcal{P}' \times \Gamma'$ does not contain a copy of $K_{2,2}$. We apply our point-curve incidence result in $\mathbb{R}^2$ (Theorem 2.1 of Chapter 3), although we need to be careful about a couple of minor issues. First, the varieties of Γ' might not be irreducible. Since each variety of Γ' is of degree $O_r(1)$, we can decompose each curve into $O_r(1)$ irreducible curves, obtaining a total of $O_r(n)$ curves. Another issue is that some of the elements of Γ might be zero-dimensional. This is not a real issue, since the proof of the planar incidence theorem remains valid also when some of the varieties are zero-dimensional. Thus, we have

$$I(\mathcal{P}_0, \mathcal{H} \setminus \mathcal{H}_0) = O_r\left(m^{2/3}n^{2/3} + m_0 + n\right).$$

As before, by using (8) and taking α_1 and α_2 to be sufficiently large with respect to r and the constant of the $O(\cdot)$-notation, we have

$$I(\mathcal{P}_0, \mathcal{H} \setminus \mathcal{H}_0) \leq \frac{\alpha_1}{3} m^{2/3}n^{2/3} + \frac{\alpha_2}{2} m_0. \tag{13}$$

Bounding $I(\mathcal{P}_0, \mathcal{H}_0)$. As in previous proofs, we consider separately singular and regular points of $Z(f)$. Consider a point $p \in \mathcal{P}_0$ such that p is incident to two 2-flats $h, h' \in \mathcal{H}_0$. We perform a translation of $\mathbb{R}^4$ so that p becomes the origin. We can then think of h and h' as vector subspaces. By the assumption of the theorem, h and h' intersect only in p, which in turn implies that the vector spaces h and h' span all of $\mathbb{R}^4$ together. However, since both h and h' are fully contained in $Z(f)$, their tangent 2-flats at p are fully contained in the tangent hyperplane to $Z(f)$ at p, which is impossible. That is, the tangent to $Z(f)$ at p is not well defined, so p is a singular point of $Z(f)$.

The above implies that at most one plane of $\mathcal{H}_0$ can be incident to a point of $\mathcal{P}_0$ that is a regular point of $Z(f)$. That is, such regular points contribute $O(m_0)$ incidences to $I(\mathcal{P}_0, \mathcal{H}_0)$.

To handle the singular points, we denote by Z_{sing} the set singular points of $Z(f)$. By Theorem 4.1 of Chapter 2, the set Z_{sing} is of dimension at most two and of degree $O_r(1)$. Thus, Z_{sing} fully contains $O_r(1)$ 2-flats of $\mathcal{H}_0$ and these yield $O_r(m_0)$ incidences with the points of $\mathcal{P}_0$. The 2-flats of $\mathcal{H}_0$ that are not fully contained in Z_{sing} intersect Z_{sing} in varieties that are at most one-dimensional and of degree $O_r(1)$. These can be handled by projecting them onto a two-dimensional plane, just as we did in the case of $I(\mathcal{P}_0, \mathcal{H} \setminus \mathcal{H}_0)$. As before, this leads to $O_r\left(m^{2/3}n^{2/3} + m_0 + n\right)$ incidences. By combining the singular and regular cases, we get

$$I(\mathcal{P}_0, \mathcal{H}_0) = O_r\left(m^{2/3}n^{2/3} + m_0 + n\right).$$

Once again, by using (8) and taking α_1 and α_2 to be sufficiently large with respect to r and the constant of the $O(\cdot)$-notation, we have

$$I(\mathcal{P}_0, \mathcal{H}_0) \leq \frac{\alpha_1}{3} m^{2/3}n^{2/3} + \frac{\alpha_2}{2} m_0. \tag{14}$$

The induction step is obtained by combining (12), (13), and (14), and this in turn completes the proof of the theorem. $\square$

It is not easy to remove the ε from the exponent in the bound of Theorem 3.2, and Zahl [6] managed to do this by using a rather complicated analysis. An alternative proof that does not rely on the polynomial method and leads to a bound with no ε was obtained by Tóth [5]. It is also not easy to extend the above analysis to general complex curves (although the specific case of complex unit circles can be handled by adding one extra combinatorial trick; see [4]). A significantly more involved analysis that extends the above proof to general complex curves was derived by Sheffer and Zahl [3].

Solymosi and Tao proved Theorem 3.2 by deriving a more general bound, using a proof that goes along the same lines as the one presented above.

Theorem 3.4 (Solymosi and Tao [4]). *Let $\mathcal{P}$ be a set of m points and let $\mathcal{V}$ denote a set of varieties of degree at most k,*[4] *both in $\mathbb{R}^d$, such that*

- *The dimension of every variety of $\mathcal{V}$ is at most $d/2$.*
- *The incidence graph contains no copy of $K_{2,t}$.*
- *There are no incidences between a point $p \in \mathcal{P}$ and a variety $U \in \mathcal{V}$ where p is a singular point of U.*
- *If a point $p \in \mathcal{P}$ is incident to two varieties $U, W \in \mathcal{V}$ and p is a regular point of both U and W, then the tangents to U and W at p intersect only in p (this property is sometimes called* transversality*).*

Then $I(\mathcal{P}, \mathcal{V}) = O_{\varepsilon,k,d,t}(m^{2/3+\varepsilon}n^{2/3} + m + n)$.

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[4] In [4], a different definition of degree is used. In the above proof we slightly changed the proof of [4] to fit our definition.