Note: Bring your solutions either to my office Sloan 264 or drop them in my mailbox but don’t put them in the Ma160 mailbox.

1) Let $K_p$ be a local field containing a primitive $l$-th root of unity where $l$ is a prime number and recall the Hilbert symbol
\[
\left( \frac{\alpha, \beta}{p} \right) := \rho_p(\beta) \in \mu_l
\]
where
\[
\rho_p : K_p^\times \to \text{Gal}(K_p(\sqrt[l]{\alpha})/K_p) \subseteq \mu_l
\]
is the local reciprocity map for the extension $K_p(\sqrt[l]{\alpha})/K_p$ followed by the map $\sigma \mapsto (\sqrt[l]{\alpha})^{\sigma - 1}$ from Kummer theory. It is a bilinear map
\[
K_p^\times/(K_p^\times)^l \times K_p^\times/(K_p^\times)^l \to \mu_l.
\]

a) Show that
\[
\left( \frac{\alpha, \beta}{p} \right) = 1
\]
if $\alpha + \beta \in (K_p^\times)^l$ (Hint: Show that $\gamma^l - \alpha$ is a norm from $K_p(\sqrt[l]{\alpha})$ for any $\gamma$). Deduce that
\[
\left( \frac{\alpha, -\alpha}{p} \right) = \left( \frac{\alpha, 1 - \alpha}{p} \right) = 1.
\]

b) Show that the Hilbert symbol is anti-symmetric, i.e.
\[
\left( \frac{\alpha, \beta}{p} \right) = \left( \frac{\beta, \alpha}{p} \right)^{-1}
\]
by evaluating $\left( \frac{\alpha, -\alpha \beta}{p} \right)$.

2) Let $K$ be a number field containing a primitive third root of unity. For $\alpha \in O_K$ and a prime $p \nmid 3\alpha$ define the cubic residue symbol (which just for this exercise we denote like the quadratic residue symbol)
\[
\left( \frac{\alpha}{p} \right) \in \mu_3
\]
as the unique third root of unity congruent to $\alpha^{p^{l-1}}$ modulo $p$. If $\beta \in O_K$ is prime to $3\alpha$ with prime factorization $(\beta) = p_1 \cdots p_r$ define
\[
\left( \frac{\alpha}{\beta} \right) = \left( \frac{\alpha}{p_1} \right) \cdots \left( \frac{\alpha}{p_r} \right).
\]
a) Show that \( \left( \frac{\alpha}{p} \right) = 1 \) if and only if \( \alpha \) is a third power modulo \( p \)

b) For \( \alpha \in O_K \) and for each prime \( p \) of \( K \) let \( \rho_p \) be the map (1) for \( l = 3 \). For \( \beta \) prime to \( 3 \alpha \) show that
\[
\prod_{p|\alpha} \rho_p(\beta) = \prod_{p|\beta} \rho_p(\beta) = \left( \frac{\alpha}{\beta} \right).
\]

c) For \( \beta \) as in b) and \( \alpha \) prime to 3 show
\[
\prod_{p|\alpha, p \nmid 3} \rho_p(\beta) = \left( \frac{\beta}{\alpha} \right)^{-1}
\]
and deduce the cubic reciprocity law
\[
\left( \frac{\alpha}{\beta} \right) \left( \frac{\beta}{\alpha} \right)^{-1} = \prod_{p|3} \rho_p(\beta)^{-1}.
\]

c) To complete the reciprocity law one needs to compute the Hilbert symbol for \( K_p \) for \( p \mid 3 \). Assume now \( K_p \cong \mathbb{Q}_3(\zeta_3) \) and let \( \pi = 1 - \zeta_3 \) be a uniformizer. Show that
\[
K_p^\times/(K_p^\times)^3 \cong \mathbb{Z}/3 \times \{ 1 \} \times \{ 2 \} \times \{ 2 - 3\zeta_3 \} \times \mathbb{Z}/3
\]
and
\[
\left( \frac{2, 2 - 3\zeta_3}{p} \right) = 1.
\]
Deduce that if \( \alpha, \beta \in O_K \) are congruent to \( \pm 1 \) modulo 3 and prime to each other, and if \( (\pi) \) splits completely in \( K/\mathbb{Q}(\zeta_3) \), we have
\[
\left( \frac{\alpha}{\beta} \right) = \left( \frac{\beta}{\alpha} \right).
\]

d) (Optional) If you are ambitious compute the full Hilbert symbol for \( K_p \cong \mathbb{Q}_3(\zeta_3) \) (there are 6 independent entries since it is a symplectic form on a four dimensional space over \( \mathbb{Z}/3 \)). Deduce supplementary laws computing \( \left( \frac{\alpha}{p} \right) \) if \( \alpha \) is a unit or 3-primary under the assumption that \( (\pi) \) splits completely in \( K/\mathbb{Q}(\zeta_3) \).

3) Prove Lemma 4.1 from class: If \( X \) is a cohomologically trivial module over a finite group \( \Gamma \) and \( A \) is any (finitely generated) \( \mathbb{Z} \)-free \( \Gamma \)-module then \( \text{Hom}_\mathbb{Z}(A, X) \) is cohomologically trivial.