In this note, we clarify a point about uniqueness of classifying spaces.
Denote by $\textbf{PW}'$ the homotopy category of pointed CW spaces, and $\textbf{PSet}$ the category of pointed sets. The contravariant functor

$$\text{Vect}^n : \textbf{PW}' \to \textbf{PSet}^{op}$$

takes any pointed CW space $X$ to the pointed set $\text{Vect}^n(X)$ of isomorphism classes of rank-$n$ vector bundles over $X$, based at the trivial bundle over $X$.

Every pointed topological space $B$ gives rise to the homotopy functor

$$\langle -, B \rangle : \textbf{PW}' \to \textbf{PSet}^{op}$$

takes any pointed CW space $X$ to the pointed set $\langle X, B \rangle$, of which the base element is the map of $X$ to the base point of $B$. Then by saying that the functor $\text{Vect}^n$ is representable by $B$, we mean that there exists a natural isomorphism of functors:

$$T : \text{Vect}^n \to \langle -, B \rangle.$$

More precisely, for any CW space $X$, there is a bijective map $T_X : \text{Vect}^n(X) \to \langle X, B \rangle$ such that for any map $f : Y \to X$ between CW spaces, the following diagram commutes:

$$\begin{array}{ccc}
\text{Vect}^n(X) & \xrightarrow{f^*} & \text{Vect}^n(Y) \\
T_X \downarrow & & \downarrow T_Y \\
\langle X, B \rangle & \xrightarrow{f^*} & \langle Y, B \rangle
\end{array}$$

**Proposition 1.** Suppose that $\text{Vect}^n$ is representable by a pointed connected topological space $B$. Then $\text{Vect}^n$ is representable by a pointed connected CW space $B'$ if and only if there exists a weak homotopy equivalence $h : B' \to B$.

**Proof.** We first show the ‘only if’ direction. Let $T$ and $T'$ be associated natural isomorphisms to $B$ and $B'$ respectively. Then $\langle -, B' \rangle$ is naturally isomorphic to $\langle -, B \rangle$ via the functor $T(T')^{-1}$, which we rewrite as $F$. In particular, plugging in $B'$, there is a map $h : B' \to B$ given by $F_B'(\text{Id}_{B'})$. Moreover, for any map $f : X \to B'$ of a pointed CW space $X$, we have a commutative diagram:

$$\begin{array}{ccc}
\langle B', B' \rangle & \xrightarrow{f^*} & \langle X, B' \rangle \\
F_{B'} \downarrow & & \downarrow F_X \\
\langle B', B \rangle & \xrightarrow{f^*} & \langle X, B \rangle
\end{array}$$

where $f^*$ is defined by pre-composition. At $\text{Id}_{B'} \in \langle B', B' \rangle$, the commutative diagram implies $F_X(f) \simeq h \circ f$. Since $f$ is arbitrary, we see that $F_X$ coincides with the operation of post-composition by $h$:

$$h : \langle X, B' \rangle \to \langle X, B \rangle.$$
Taking $X$ to be $S^m$, we conclude that $h : B' \to B$ induces isomorphism on $\pi_m$ as $F_X$ is bijective. Therefore, $h : B' \to B$ is a weak homotopy equivalence.

It remains to show the ‘if’ direction. Suppose that $h : B' \to B$ is a weak homotopy equivalence where $B'$ is CW. By Hatcher [AT, Proposition 4.22], the induced map $h_* : \langle X, B' \rangle \to \langle X, B \rangle$, is bijective for all pointed CW space $X$, so we can take $T_X'$ to be $(h_2)^{-1} \circ T_X$. \hfill $\square$

Therefore, if $\text{Vect}^n$ is representable by a pointed connected topological space $B$, we can perform a CW approximation and assume in addition that $B$ is CW. In this case, $\text{Vect}^n(B)$ contains a distinguished element $T_B^{-1}(\text{Id}_B)$, which we denote by $\gamma_n$. The bundle $\gamma_n$ is called a universal rank-$n$ vector bundle over a classifying space $B$. Then it is easy to see that $B$ is unique up to homotopy equivalence and $\gamma_n$ is unique up to bundle equivalence.

If we started by assuming that $B$ is CW, a universal property argument works for the uniqueness, as Dmitri correctly pointed out in class.