Problem 1
The following is a direct generalization of Fatou’s theorem: Show that if \( u(re^{i\theta}) \) is harmonic in the unit disc and bounded there, then \( \lim_{r \to 1} u(re^{i\theta}) \) exists for a.e. \( \theta \).

Problem 2
Show that at almost every point of the boundary unit circle, the function \( \sum_{n=0}^{\infty} z^{2^n} \) fails to have radial limit. This is an example of function that fails to have radial limits almost everywhere.

Problem 3
Suppose \( F \) is holomorphic in the unit disc, and
\[
\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{i\theta})| d\theta < \infty,
\]
where \( \log^+ u = \log u \) if \( u \geq 1 \), and \( \log^+ u = 0 \) if \( u < 1 \).
Then \( \lim_{r \to 1} F(re^{i\theta}) \) exists for almost every \( \theta \). The above condition is satisfied whenever
\[
\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta < \infty, \quad \text{for some } p > 0
\]
Functions that satisfy the latter condition are said to belong to the Hardy space \( H^p(D) \).

Problem 4
Let \( F(z) \) be a bounded holomorphic function in the upper half-plane. Show that \( \lim_{y \to 0} F(x + iy) \) exists for a.e. \( x \).

Problem 5
Consider \( F(z) = \frac{1}{z+1}e^z \) in the upper half-plane. Note that \( F(x + iy) \in L^2(\mathbb{R}) \), for each \( y > 0 \) and \( y = 0 \). Observe also that \( F(z) \to 0 \) as \( |z| \to 0 \). However, \( F \notin H^2(\mathbb{R}_+^2) \). Why?

Problem 6
Let \( H \) be the Hilbert transform. Verify that
\begin{enumerate}
  \item \( H^* = -H \), \( H^2 = -I \), and \( H \) is unitary.
\end{enumerate}
(b) If \( \tau_h \) denotes the translation operator, \( \tau_h(f)(x) = f(x - h) \), then \( H \) commutes with \( \tau_h \), \( \tau_h H = H \tau_h \).

(c) If \( \delta_a \) denotes the dilation operator, \( \delta_a(f)(x) = f(ax) \) with \( a > 0 \), then \( H \) commutes with \( \delta_a \), \( \delta_a H = H \delta_a \).

(d) Show that a bounded operator on \( L^2(\mathbb{R}^d) \) commutes with translations if and only if it is a Fourier multiplier operator.

**Problem 7**

Let \( f \in L^2(\mathbb{R}) \) and let \( u(x, y) \) be the Poisson integral of \( f \), that is \( u = (f * \mathcal{P}_y)(x) \), where \( \mathcal{P}_y \) is the Poisson kernel on the upper half-plane. Let \( v(x, y) = (Hf * \mathcal{P}_y)(x) \), the Poisson integral of the Hilbert transform of \( f \). Prove that:

(a) \( F(x + iy) = u(x, y) + iv(x, y) \) is analytic in the half-plane \( \mathbb{R}_2^+ \), so that \( u \) and \( v \) are conjugate harmonic functions. We also have \( f = \lim_{y \to 0} u(x, y) \) and \( Hf = \lim_{y \to 0} v(x, y) \).

(b) \( F(z) = \frac{1}{\pi i} \int_{\mathbb{R}} f(t) \frac{dt}{t - z} \).

(c) \( v(x, y) = f * Q_y \), where \( Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2} \) is the conjugate Poisson kernel.

**Problem 8**

Show that

\[
\left\{ \frac{1}{\sqrt{\pi(i+z)}} \left( \frac{i - z}{i + z} \right)^n \right\}_{n=0}^\infty
\]

is an orthonormal basis of \( H^2(\mathbb{R}_2^+) \).