Problem 1
Recall that we defined the $p,w$ “norm” by

$$\|f\|_{p,w} = \left(\sup_t t^p \mu\{x \mid |f(x)| > t\}\right)^{1/p}$$

which was not a norm. Define for $1 < p < \infty$ the Calderón norm by

$$\|f\|_{p,c} = \sup_{A \subset X} \mu(A)^{-1+\frac{1}{p}} \int_A |f(x)| \, d\mu(X)$$

(a) Prove that $\|\cdot\|_{p,c}$ is a norm.
(b) Prove that

$$\|f\|_{p,w} \leq \|f\|_{p,c} \leq \frac{p}{p-1} \|f\|_{p,w}$$

Problem 2
If $1 \leq p_1 < p < p_2 < \infty$ and $f \in L^{p_1}_w \cap L^{p_2}_w$, then $f \in L^p$, then from class we know

$$\|f\|_p \leq \left(\frac{p}{p_2 - p}\right) \|f\|_{p_2,w}^{\frac{p}{p_2 - p}} + \left(\frac{p}{p - p_1}\right) \|f\|_{p_1,w}^{\frac{p}{p - p_1}}.$$

Improve this inequality to

$$\|f\|_p \leq A \|f\|_{p_1,w}^{\alpha} \|f\|_{p_2,w}^{1-\alpha},$$

where

$$A = \left(\frac{p}{p-p_1} + \frac{p}{p_2 - p}\right)^{1/p}, \quad \alpha = \frac{p^{-1} - p_{2}^{-1}}{p_1^{-1} - p_2^{-1}}.$$

Problem 3
We look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x,y) : 0 \leq x \leq \pi, 0 \leq y \leq 1\}$ that vanishes on the vertical sides of $R$, and so that

$$u(x,0) = f_0(x) \quad \text{and} \quad u(x,1) = f_1(x)$$

where $f_0$ and $f_1$ are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if $f_0$ and $f_1$ have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \quad \text{and} \quad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$
then
\[ u(x, y) = \sum_{k=1}^{\infty} \left( \frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx. \]

We recall the definitions of the hyperbolic sine and cosine functions:
\[ \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}. \]

Compare this result with the solution of the Dirichlet problem in the strip.

**Problem 4**

If \( P_r(\theta) \) denotes the Poisson kernel for the disc, show that the function
\[ u(r, \theta) = \frac{\partial P_r}{\partial \theta}, \]
defined for \( 0 \leq r < 1 \) and \( \theta \in \mathbb{R} \), satisfies:

(i) \( \Delta u = 0 \) in the disc.

(ii) \( \lim_{r \to 1} u(r, \theta) = 0 \) for each \( \theta \).

However, \( u \) is not identically zero.

**Problem 5**

Solve Laplace’s equation \( \Delta u = 0 \) in the semi infinite strip
\[ S = \{(x, y) : 0 < x < 1, 0 < y \}, \]
subject to the following boundary conditions
\[
\begin{align*}
  u(0, y) &= 0 \quad \text{when } 0 \leq y, \\
  u(1, y) &= 0 \quad \text{when } 0 \leq y, \\
  u(x, 0) &= f(x) \quad \text{when } 0 \leq x \leq 1
\end{align*}
\]
where \( f \) is a given function, with of course \( f(0) = f(1) = 0 \). Write
\[ f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \]
and expand the general solution in terms of the special solutions given by
\[ u_n(x, y) = e^{-n\pi y} \sin(n\pi x) \]
Express \( u \) as an integral involving \( f \), analogous to the Poisson integral formula.

**Problem 6**

Consider the Dirichlet problem in the annulus defined by \((r, \theta) : \rho < r < 1\), where \( 0 < \rho < 1 \) in the inner radius. The problem is to solve
\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]
subject to the boundary conditions

\[
\begin{align*}
  u(1, \theta) &= f(\theta) \\
  u(\rho, \theta) &= g(\theta)
\end{align*}
\]

where \( f \) and \( g \) are given continuous functions.

Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

\[
u(r, \theta) = \sum c_n(r) e^{in\theta}
\]

with \( c_n(r) = A_n r^n + B_n r^{-n} \), \( n \neq 0 \). Set

\[
f(\theta) \sim \sum a_n e^{in\theta} \quad \text{and} \quad g(\theta) \sim \sum b_n e^{in\theta}
\]

We want \( c_n(1) = a_n \) and \( c_n(\rho) = b_n \). This leads to the solution

\[
u(r, \theta) = \sum_{n \neq 0} \left( \frac{1}{\rho^n - r^n} \right) \left[ \left( \left( \frac{\rho}{r} \right)^n - \left( \frac{r}{\rho} \right)^n \right) a_n + (r^n - r^{-n}) b_n \right] e^{in\theta} + a_0 + (b_0 + a_0) \frac{\log r}{\log \rho}
\]

Show that as a result we have

\[
u(r, \theta) - (P_r \ast f)(\theta) \to 0 \quad \text{as} \quad r \to 1 \quad \text{uniformly in} \quad \theta,
\]

and

\[
u(r, \theta) - (P_\rho/r \ast g)(\theta) \to 0 \quad \text{as} \quad r \to \rho \quad \text{uniformly in} \quad \theta.
\]

**Problem 7**

The equation

\[
x^2 \frac{\partial^2 u}{\partial x^2} + ax \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}
\]

with \( u(x, 0) = f(x) \) for \( 0 < x < \infty \) and \( t > 0 \) is a variant of the heat equation which occurs in a number of applications. To solve (1), make the change of variables \( x = e^{-y} \) so that \( -\infty < y < \infty \). Set \( U(y, t) = u(e^{-y}, t) \) and \( F(y) = f(e^{-y}) \). Then the problem reduces to the equation

\[
\frac{\partial^2 U}{\partial y^2} + (1 - \alpha) \frac{\partial U}{\partial y} = \frac{\partial U}{\partial t}
\]

with \( U(y, 0) = F(y) \). This can be solved like the usual heat equation (the case \( \alpha = 1 \)) by taking the Fourier transform in the \( y \) variable. One must then compute the integral \( \int_{-\infty}^{\infty} e^{(-4 \pi^2 \xi^2 + (1-a)^2 \pi \xi \xi)} e^{2\pi i \xi v} d\xi \). Show that the solution of the original problem is then given by

\[
u(x, t) = \frac{1}{(4\pi t)^{1/2}} \int_{0}^{\infty} e^{-(\log(v/x)+(1-a)t)^2/(4t)} f(v) \frac{dv}{v}.
\]

**Problem 8**

Let \( F \) be a continuous function on the close \( \mathbb{D} \) of the unit disc. Assume that \( F \) is in \( C^1 \) on the (open) disc \( \mathbb{D} \), and \( \int_{\mathbb{D}} \left| \nabla F \right|^2 < \infty \).

Let \( f(e^{i\theta}) \) denote the restriction of \( F \) to the unit circle, and write \( f(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \). Prove that \( \sum_{n=-\infty}^{\infty} |n| |a_n|^2 < \infty \).