Problem 1

Prove that if \( f \) is integrable on \( \mathbb{R}^d \), and \( f \) is not identically zero, then
\[
f^*(x) \geq \frac{c}{|x|^d}, \quad \text{for some } c > 0 \text{ and all } |x| \geq 1.
\]

Conclude that \( f^* \) is not integrable on \( \mathbb{R}^d \). Then, show that the weak type estimate
\[
m\left(\{x : f^*(x) > \alpha\}\right) \leq \frac{c}{\alpha}
\]
for all \( \alpha > 0 \) whenever \( \int_{\mathbb{R}^d} |f| = 1 \), is best possible in the following sense: if \( f \) is supported in the unit ball with \( \int_{\mathbb{R}^d} |f| = 1 \), then
\[
m\left(\{x : f^*(x) > \alpha\}\right) \geq \frac{c'}{\alpha}
\]
for some \( c' > 0 \) and all sufficiently small \( \alpha \).

Problem 2

Consider the function on \( \mathbb{R} \) defined by
\[
f(x) = \begin{cases} \frac{1}{|x|^d \log \frac{1}{|x|^d}} & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}
\]

(a) Verify that \( f \) is integrable.

(b) Establish the inequality
\[
f^*(x) \geq \frac{c}{|x| \log \frac{1}{|x|}} \quad \text{for some } c > 0 \text{ and all } |x| \leq \frac{1}{2},
\]

to conclude that the maximal function \( f^* \) is not locally integrable.

Problem 3

Prove that if \( \{K_\epsilon\}_{\epsilon > 0} \) is a family of approximations to the identity, then
\[
\sup_{\epsilon > 0} |(f * K_\epsilon)(x)| \leq c f^*(x)
\]
for some constant \( c > 0 \) and all integrable \( f \).

Recall that a family of functions \( \{K_\epsilon\}_{\epsilon > 0} \) is called a family of approximations to the identity if (i) \( \int_{\mathbb{R}^d} K_\epsilon = 1 \), (ii) \( |K_\epsilon(x)| \leq A \epsilon^{-d} \) for all \( \epsilon > 0 \), (iii) \( |K_\epsilon(x)| \leq A \epsilon/|x|^{d+1} \) for all \( \epsilon > 0, x \in \mathbb{R}^d \).
Problem 4

Let $\mathcal{R}$ denote the set of all rectangles in $\mathbb{R}^2$ that contain the origin, and with sides parallel to the coordinate axis. Consider the maximal operator associated to this family, namely

$$f^*_R(x) = \sup_{R \in \mathcal{R}} \frac{1}{m(R)} \int_R |f(x-y)|dy.$$  

(a) Then, $f \mapsto f^*_R \cdot$ does not satisfy the weak type inequality

$$m\{x : f^*_R(x) > \alpha\} \leq \frac{A}{\alpha} \|f\|_{L^1}$$

for all $\alpha > 0$, all integrable $f$, and some $A > 0$.

(b) Using this, one can show that there exists $f \in L^1(\mathbb{R})$ so that for $R \in \mathcal{R}$

$$\limsup_{\text{diam}(R) \to 0} \frac{1}{m(R)} \int_R f(x-y)dy = \infty \quad \text{for almost every } x.$$  

Here $\text{diam}(R) = \sup_{x,y \in R} |x - y|$ equals the diameter of the rectangle.

Problem 5

Suppose $\tau$ is a measure-preserving transformation on a measure space $(X, \mu)$ with $\mu(X) = 1$. Recall that a measurable set $E$ is invariant if $\tau^{-1}(E)$ and $E$ differ by a set of measure zero. A sharper notion is to require that $\tau^{-1}(E)$ equal $E$. Prove that if $E$ is any invariant set, there is a set $E'$ so that $E' = \tau^{-1}(E')$, and $E$ and $E'$ differ by a set of measure zero.

Problem 6

Let $\tau$ be a measure-preserving transformation on $(X, \mu)$ with $\mu(X) = 1$. Then $\tau$ is ergodic if and only if whenever $\nu$ is absolutely continuous with respect to $\mu$ and $\nu$ is invariant (that is, $\nu(\tau^{-1}(E)) = \nu(E)$ for all measurable sets $E$), then $\nu = c\mu$, with $c$ a constant.

Problem 7

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the torus, and $\tau : x \mapsto x + \alpha$ where $\alpha \in \mathbb{R}^d$. Then $\tau$ is ergodic if and only if $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_1, \alpha_2, \cdots, \alpha_d, \text{ and } 1$ are linearly independent over the rationals. To do this show that:

(a) $\frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k(x)) \to \int_{\mathbb{T}^d} f(x)dx$ as $m \to \infty$, for each $x \in \mathbb{T}^d$, whenever $f$ is continuous and periodic and $\alpha$ satisfies the hypothesis.

(b) Prove as a result that in this case $\tau$ is uniquely ergodic.

Problem 8

Let $X = \prod_{i=1}^\infty X_i$, where each $(X_i, \mu_i)$ is identical to $(X_1, \mu_1)$, with $\mu_1(X_1) = 1$, and let $\mu$ be the corresponding product measure defined on the sets $\prod_{i=1}^\infty E_i$ where $E_i \subseteq X_i$ are measurable and only finitely many $E_i$ differ from $X_i$. Define this shift $\tau : X \to X$ by $\tau((x_1, x_2, \cdots)) = (x_2, x_3, \cdots)$ for $x = (x_i) \in \prod_{i=1}^\infty X_i$.

(a) Verify that $\tau$ is a measure-preserving transformation.

(b) Prove that $\tau$ is ergodic by showing that it is mixing.
(c) Note that in general $\tau$ is not uniquely ergodic.

(d) If we define the corresponding shift on the two-sided infinite product, then $\tau$ is also a measure-preserving isomorphism.

**Problem 9**

Consider an automorphism $A$ of $T^d$, that is, $A$ is a linear isomorphism of $\mathbb{R}^d$ that preserves the lattice $\mathbb{Z}^d$. Note that $A$ can be written as a $d \times d$ matrix whose entries are integers, with $\det A = \pm 1$. Define the mapping $\tau : T^d \to T^d$ by $\tau(x) = A(x)$.

1. Observe that $\tau$ is a measure-preserving isomorphism of $T^d$.

2. Show that $\tau$ is ergodic (in fact, mixing) if and only if $A$ has no eigenvalues of the form $e^{2\pi ip/q}$, where $p$ and $q$ are integers.

3. Note that $\tau$ is never uniquely ergodic.

**Problem 10**

Let $X = [0,1)$, $\tau(x) = <1/x>$, $x \neq 0$, $\tau(0) = 0$. Here $<x>$ denotes the fractional part of $x$. With the measure $d\mu = \frac{1}{\log 2} \frac{dx}{1+x}$, we have of course $\mu(X) = 1$.

(a) Show that $\tau$ is a measure-preserving transformation.

(b) Show that $\tau$ is ergodic.