Homework 6: Due May 22th 11am, Friday, 2015

1. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain. Let $\{f_j\}$ be a sequence of holomorphic functions $\Omega \to \mathbb{C}$ such that there exists a uniform bound $C < \infty$ as in
$$\int_{\Omega} |f_j(z)|^2 \, dx \, dy < C.$$ 
Show that $\{f_j\}$ form a normal family. [Hint: Use the Cauchy inequality to show that $|f(z)|^2$ does not exceed the mean value of $|f|^2$ on a small disc around $z$ (see Greene-Krantz: Exercise 8 in Chapter 4). Then deduce that $f_j$ are locally uniformly bounded (i.e. bounded on compact subsets).]

2. Use the open mapping principle for holomorphic functions to prove an open mapping principle for real-valued harmonic functions.

3. Compute a formula analogous to the Poisson integral formula, for the region $\mathbb{C}_+ = \{ z : \text{Im}(z) > 0 \}$, by mapping $\mathbb{C}_+$ conformally to the unit disk.

4. Suppose $\Omega$ is a bounded region. Let $L$ be a (two-way infinite) line that intersects $\Omega$. Assume that $\Omega \cap L$ is an interval $I$. Choosing an orientation for $L$, we can define $\Omega_\ell$ and $\Omega_r$ to be subregions of $\Omega$ lying strictly to the left or right of $L$, with $\Omega = \Omega_\ell \cup I \cup \Omega_r$ a disjoint union. Show that if $\Omega_\ell$ and $\Omega_r$ are simply connected, then $\Omega$ is simply connected.

5. Let
$$g(z) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{h(x)}{x - z} \, dx$$
where $h$ is continuous and supported in $[-M,M]$.
   a) Prove that the function $g$ is holomorphic in $\mathbb{C} \setminus [-M,M]$, and vanishes at infinity, that is, $\lim_{|z| \to \infty} |g(z)| = 0$. Moreover, the ”jump” of $g$ across $[-M,M]$ is $h$, that is,
$$h(x) = \lim_{\varepsilon \to 0, \varepsilon > 0} g(x + i\varepsilon) - g(x - i\varepsilon).$$
   Hint: Express the difference $g(x + i\varepsilon) - g(x - i\varepsilon)$ in terms of a convolution of $h$ with the Poisson kernel.
   b) If $h$ satisfies a mild smoothness condition, for instance a Hölder condition with exponent $\alpha$, that is, $|h(x) - h(y)| \leq C|x - y|^\alpha$ for some $C > 0$ and all $x, y \in [-M,M]$, then $g(x + i\varepsilon)$ and $g(x - i\varepsilon)$ converge uniformly to functions $g_+(x)$ and $g_-(x)$ as $\varepsilon \to 0$. Prove that $g$ can be characterized as the unique holomorphic function that satisfies:
   (i) $g$ is holomorphic outside $[-M,M]$,
(ii) $g$ vanishes at infinity,
(iii) $g(x + i\varepsilon)$ and $g(x - i\varepsilon)$ converge uniformly as $\varepsilon \to 0$ to functions $g_+(x)$ and $g_-(x)$ with
\[ g_+(x) - g_-(x) = h(x). \]