1. Assignment 1

(due on Tuesday, April 7 by 2PM)

Policy: You may collaborate with your classmates and you are allowed to consult books while solving the homework problems. However you should not consult any other people (except the instructor and TA) or use internet resources such as online forums. Please cite book references if any are used. If you use results not proved in class please provide your own proof.

1. Assignment 1

(a) Is \( \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}} \rangle \) algebraic? Explain why or why not.

(b) Let

\[ e = \sum_{n=0}^{\infty} \frac{1}{n!} \]

Show that \( e \) is irrational. (Hint: Suppose \( e = \frac{a}{b} \). Consider \( b^e \) and prove it cannot be an integer)

2.

(a) Prove by induction for every integer \( m \geq 2 \), the property

\[ P(m) : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(m-1)m} = \frac{m-1}{m} \]

(b) Prove (\( * \)) by using well ordering instead of induction.

(c) Use (\( * \)) to deduce the following (for any \( m \geq 2 \)):

\[ 1 < \sum_{n=1}^{m} \frac{1}{n^2} < 2 \]

Conclude by taking the limit as \( m \to \infty \) that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \in [1, 2] \]

3. Give an example of a binary relation on \( \mathbb{N} \) which is

(a) reflexive and transitive, but not symmetric

(b) transitive and symmetric, but not reflexive

(c) reflexive and symmetric, but not transitive

(d) antisymmetric and transitive, but not reflexive.

4.

(a) Given \( n + 1 \) distinct positive integers \( x_1, \cdots, x_{n+1} \) less than or equal to \( 2n \), prove that there exists \( i \neq j \) such that \( x_i | x_j \).
(b) Given \( n \) integers \( x_1, \cdots, x_n \) not necessarily distinct, show that there exist integers \( 1 \leq l \leq m \leq n \) such that

\[
x_l + x_{l+1} + \cdots + x_m
\]

is a multiple of \( n \).

5.

(a) Show that if \( \alpha \in \mathbb{R} \) and \( n \in \mathbb{Z}_+ \) there exists \( a, b \in \mathbb{Z} \) with \( 1 \leq a \leq n \) such that

\[
|a\alpha - b| \leq \frac{1}{n + 1}
\]

(Hint: Improve slightly the proof of Dirichlet’s approximation theorem proved in class to get this stronger result)

(b) Use Dirichlet’s approximation theorem to show that if \( \alpha \) is irrational then there exist infinitely many positive integers \( q \) such that there is an integer \( p \) such that

\[
|\alpha - p/q| \leq \frac{1}{q^2}.
\]

(Hint: Could you have a fixed pair \((a,b)\) satisfying the inequality in (a) for infinitely many values of \( n \in \mathbb{Z}_+ \)?)