# Homework 8: Due Tuesday May 26, 1pm

1. (10 pts.) Consider the theory of linear orders: $T =$. Show that there is no theory $T' \supseteq T$ whose models are exactly the wellorders (recall that a wellorder is a linear order such that there is no infinite descending sequence $a_1 > a_2 > a_3 > \ldots$). [Hint: add infinitely many constants $c_1, c_2, \ldots$ to the language. Then for each $n$, construct a sentence $\phi_n$ stating that the first $n$ of these constants form a descending sequence. Then use compactness.]

2. (10 pts., no collab) Consider the theory $T$ of graphs. Show that there is no theory $T' \supseteq T$ whose models are exactly the disconnected graphs; i.e. the graphs so that there are two vertices with no path between them.

3. (10 pts.) Recall our two versions of the compactness theorem.

Version 1 Suppose $L$ is a language, $T$ is a theory in $L$, and $\phi$ is a sentence in $L$. If $T \vDash \phi$, then there is a finite subset $T_0 \subseteq T$ such that $T_0 \vDash \phi$.

Version 2 Suppose $L$ is a language, $T$ is a theory in $L$. Then $T$ is satisfiable iff every finite subset $T_0$ is satisfiable.

Without assuming the completeness theorem or any of its consequences we have derived, use version 2 of the compactness theorem to prove version 1.

4. (20 pts.) Suppose that $p(x)$ and $q(x)$ are nonzero polynomials and $(p_i(x))_{0 \leq i \leq n}$ is the sequence

$$\begin{aligned}
p_0(x) &= p(x) \\
p_1(x) &= p'(x)q(x) \\
p_2(x) &= -\operatorname{remainder}(p_0(x), p_1(x)) \\
p_3(x) &= -\operatorname{remainder}(p_1(x), p_2(x)) \\
&\ldots \\
p_n(x) &= -\operatorname{remainder}(p_{n-1}(x), p_{n-2}(x))
\end{aligned}$$

Let $s(-\infty)$ be the sequence giving the sign of each $p_i(x)$ as $x \to -\infty$ and $s(\infty)$ be the sequence giving the sign of each $p_i(x)$ as $x \to \infty$.

a) Suppose $p_n(x)$ is constant, so $p(x)$ does not share any roots with $p'(x)q(x)$. Then show the number of sign changes in the sequence

$s(-\infty)$ mionus the number of sign changes in the sequence $s(\infty)$ is equal to the number of roots of $p(x)$ where $q(x) > 0$ minus the number of roots of $p(x)$ where $q(x) < 0$.

b) Now show that the above is true for all $p(x)$ and $q(x)$ even when $p_n(x)$ is not constant. [Hint: let $g(x)$ be the gcd of $p(x)$ and $p'(x)q(x)$. Then divide the sequence $p_0(x), p_1(x), \ldots$ by $g(x)$.]