Homework 6: Due Tuesday May 12, 1pm

1. (15 pts) Consider the language $\mathcal{L} = \{a, +, \cdot, <\}$, where a is a constant symbol, $+, \cdot$ are binary function symbols and $<$ is a binary relation symbol, and its structure
$$\mathcal{M} = \langle M, 2, +, \cdot, < \rangle$$
where $M = \{2, 3, 4, \dots\}$ and $+, \cdot, <$ have their usual meanings. Let
$$\begin{aligned} A : \forall x \exists y \forall z \exists w \exists v \{(x < y) \wedge [z = w \cdot v \\ \vee\, w \cdot z = (x + y) \\ \vee\, w \cdot (x + y) = v \cdot z + a]\}. \end{aligned}$$
We can play $G_{\mathcal{M},A}$, slightly modified from what was described in class last week. First $\forall$ plays $a \in M$, then $\exists$ plays $b \in M$, then $\forall$ plays $c \in M$, and finally $\exists$ plays $d, e \in M$. $\exists$ wins iff $\mathcal{M} \vDash A[a, b, c, d, e]$. (Note that by adding in a few dummy variables and quantifiers, we can make this exactly of the form we discussed in class.) Determine which of the two players $\exists, \forall$ has a winning strategy for this game.

2. (15 pts) Let L be a first order language, and ϕ a sentence of L. Then the *spectrum* of ϕ is the set of possible sizes of universes of *finite* models of ϕ. That is,
$$\begin{aligned} \text{spectrum}(\phi) = \{n \geq 1 :\ & \text{there is a model } M \text{ of } \phi \\ & \text{so that the universe of } M \text{ has exactly } n \text{ elements}\} \end{aligned}$$
For each of the sets X below find a language L and sentence ϕ such that spectrum$(\phi) = X$.
   (a) $X = \emptyset$
   (b) $X = \{1, 2, 3, \dots\}$
   (c) $X = \{n \geq 1 :\ n \text{ is even}\}$.
   (d) $X = \{n \geq 1 :\ n \text{ is a square}\}$.

3. (No collab 10 pts) Suppose S is a set of first order formulas, and ϕ and ψ are first order formulas. Then show:
   (a) (The deduction theorem) $S \cup \{\phi\} \vdash \psi$ iff $S \vdash \phi \rightarrow \psi$.

(b) (Proof by contradiction) If $S \cup \{\phi\} \vdash \psi$ and $S \cup \{\phi\} \vdash \neg\psi$, then $S \vdash \neg\phi$.

4. (15 pts) Show using our formal proof system that:

   (a) $\vdash \forall x(P(x) \wedge Q(x)) \leftrightarrow (\forall x P(x) \wedge \forall x Q(x))$.

   (b) $\{P(x), \forall y(P(y) \rightarrow (\forall z Q(z)))\} \vdash \forall x Q(x)$.

   (c) $\vdash \exists x \forall y R(x, y) \rightarrow \forall y \exists x R(x, y)$