Homework 5: Due Tuesday May 5, 1pm

1. (5 pts) Write a sentence φ in the language whose only relation is $=$ such that φ is true in a structure M iff M has a universe with exactly 6 elements.

2. (5 pts) Find a sentence which is true in the structure $\langle \mathbb{Q}; < \rangle$ but not in the structure $\langle \mathbb{Z}; < \rangle$, where $\mathbb{Z}$ is the set of integers, and $\mathbb{Q}$ is the set of rational numbers.

3. (5 pts) Show that there is no structure M with a finite universe such that exactly one element of the universe of M is not definable.

4. (No collab. 10 pts) Let $S\colon \mathbb{N} \to \mathbb{N}$ be the successor function: $S(n) = n+1$. Call a set $X \subseteq \mathbb{N}$ eventually periodic if there is a p (a *period*) and an n_0 such that for all $n \geq n_0$, we have

   $$n \in X \leftrightarrow n + p \in X.$$

   Show that every eventually periodic set is definable in the structure $\langle \mathbb{N}, 0, S, + \rangle$.

5. (10 pts) Use the automorphism method to show that the set of rational numbers is not definable in $\langle \mathbb{R}; 0, 1, \cdot \rangle$.

6. (15 pts) Use the automorphism method to show that the function $+$ is not definable in the structure $\langle \mathbb{N}, \cdot \rangle$.

7. (15 pts) Prove that there are no nontrivial automorphisms of the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. (Recall that an automorphism is called nontrivial if it is not equal to the identity function $\pi(x) = x$).

8. (8 pts) Find equivalent formulas to the following that are in prenex normal form:

   (a) $\neg \forall x [\neg \big(\exists y (P(y)) \vee P(x) \vee \neg \forall z (R(x, z)) \big)]$

   (b) $\neg \forall x (P(x) \to \neg (\exists y Q(x, y) \vee \exists y R(x, y)))$.