Homework 4: Due Tuesday April 28, 1pm

1. (25 pts) Suppose S is a set of formulas of propositional logic.
   (a) Prove that $\vdash \neg\neg\phi \to \phi$. (This is 4b from last homework.)
   (b) Prove that if $S \cup \{\phi\}$ is inconsistent, then $S \vdash \neg\phi$ (that is, there is a proof of ϕ from S in our Hilbert-style proof system).
   (c) Prove that if S is inconsistent, then for every ψ, $S \vdash \psi$.
2. (15 pts) Prove that if S is a complete consistent set of formulas of propositional logic in the variables $\{p_1, p_2, \ldots\}$, then the following valuation satisfies S: assign True to p_i if the formula p_i is in S, and assign False to p_i if the formula $\neg p_i$ is in S. Show furthermore that this is the only valuation making every formula of S true.
3. (No collaboration) (12 pts) For each sentence in informal english, write a formula in first-order logic expressing it.
   (a) Every planar graph has a 4-coloring. (Our universe is the set of graphs, $P(x)$ is the relation saying the graph x is planar, and $C_4(x)$ is the relation saying that x has a 4-coloring)
   (b) Every student without a scholarship pays tuition. (Our universe is the set of students, $S(x)$ is the relation saying the student x has a scholarship, and $T(x)$ is the relation saying the student x pays tuition).
   (c) A student must complete the requirements of their major and all their core requirements to graduate. (Our universe is the set of students, $M(x)$ is the relation saying the student x has completed the requirements of their major, $C(x)$ is the relation saying the student has completed their core requirements, $G(x)$ is the relation saying the student has graduated.)
   (d) Everyone has a friend. (Our universe is the set of people. $F(x, y)$ is the relation that x is a friend of y.)
   (e) A friend of a friend is a friend. (Our universe is the set of people. $F(x, y)$ is the relation that x is a friend of y.)
   (f) There is someone with at least 3 different friends. (Our universe is the set of people. $F(x, y)$ is the relation that x is a friend of y.)

   (continued...)

4. (12 pts) For each given formula ϕ of first-order logic, give an example of a structure M in the language consisting of a single binary relation R such that $M \models \phi$, and another structure M' such that $M' \models \neg\phi$.

   (a) $\forall x[\exists y R(x, y)]$.

   (b) $\forall x[\exists y R(x, y)] \to \forall x[\forall y R(x, y)]$.

   (c) $\forall x[\forall y[R(x, y) \to \exists z R(x, z) \wedge R(z, y)]]$.

5. (12 pts) For each pair of formulas, ϕ and ψ, give an example of a structure M such that $M \models \phi$ and $M \models \neg\psi$, where our language has a binary relation R and unary relations S and T.

   (a) $\phi = \forall x \exists y R(x, y)$, $\psi = \exists y \forall x R(x, y)$.

   (b) $\phi = \forall x[S(x)] \to \forall x[T(x)]$, $\psi = \forall x[S(x) \to T(x)]$.

   (c) $\phi = \exists x[S(x)] \wedge \exists x[T(x)]$, $\psi = \exists x[S(x) \wedge T(x)]$.