1. (25 pts) Suppose $S$ is a set of formulas of propositional logic.
   (a) Prove that $\vdash \neg \neg \phi \rightarrow \phi$. (This is 4b from last homework.)
   (b) Prove that if $S \cup \{ \phi \}$ is inconsistent, then $S \vdash \neg \phi$ (that is, there is a
       proof of $\phi$ from $S$ in our Hilbert-style proof system).
   (c) Prove that if $S$ is inconsistent, then for every $\psi$, $S \vdash \psi$.

2. (15 pts) Prove that if $S$ is a complete consistent set of formulas of propositional logic in the variables
   $\{p_1, p_2, \ldots\}$, then the following valuation satisfies $S$: assign True to $p_i$ if the formula $p_i$ is in $S$, and assign False to $p_i$ if the formula $\neg p_i$ is in $S$. Show furthermore that this is the only valuation making every formula of $S$ true.

3. (No collaboration) (12 pts) For each sentence in informal english, write a
   formula in first-order logic expressing it.
   (a) Every planar graph has a 4-coloring. (Our universe is the set of
       graphs, $P(x)$ is the relation saying the graph $x$ is planar, and $C_4(x)$
       is the relation saying that $x$ has a 4-coloring)
   (b) Every student without a scholarship pays tuition. (Our universe is
       the set of students, $S(x)$ is the relation saying the student $x$ has
       a scholarship, and $T(x)$ is the relation saying the student $x$ pays
       tuition).
   (c) A student must complete the requirements of their major and all
       their core requirements to graduate. (Our universe is the set of stu-
       dents, $M(x)$ is the relation saying the student $x$ has completed the
       requirements of their major, $C(x)$ is the relation saying the student
       has completed their core requirements, $G(x)$ is the relation saying
       the student has graduated.)
   (d) Everyone has a friend. (Our universe is the set of people. $F(x,y)$ is
       the relation that $x$ is a friend of $y$.)
   (e) A friend of a friend is a friend. (Our universe is the set of people. $F(x,y)$ is the relation that $x$ is a friend of $y$.)
   (f) There is someone with at least 3 different friends. (Our universe is
       the set of people. $F(x,y)$ is the relation that $x$ is a friend of $y$.)

(continued...)
4. (12 pts) For each given formula $\phi$ of first-order logic, give an example of a structure $M$ in the language consisting of a single binary relation $R$ such that $M \models \phi$, and another structure $M'$ such that $M' \models \neg \phi$.

(a) $\forall x[\exists yR(x, y)]$.
(b) $\forall x[\exists yR(x, y)] \rightarrow \forall x[\forall yR(x, y)]$.
(c) $\forall x[\forall y[R(x, y) \rightarrow \exists zR(x, z) \land R(z, y)]]$.

5. (12 pts) For each pair of formulas, $\phi$ and $\psi$, give an example of a structure $M$ such that $M \models \phi$ and $M \models \neg \psi$, where our language has a binary relation $R$ and unary relations $S$ and $T$.

(a) $\phi = \forall x[\exists yR(x, y)], \psi = \exists y[\forall xR(x, y)]$.
(b) $\phi = \forall x[S(x)] \rightarrow \forall x[T(x)], \psi = \forall x[S(x) \rightarrow T(x)]$.
(c) $\phi = \exists x[S(x)] \land \exists x[T(x)], \psi = \exists x[S(x) \land T(x)]$. 