Homework 2: Due Tuesday April 14, 1pm

1. (10 pts) Say that a formula $\phi$ is in conjunctive normal form (CNF) if $\phi$ is of the form $\phi = \psi_1 \land \psi_2 \land \ldots \land \psi_n$, where each $\psi_i$ is of the form $\psi_i = \ell_{i,1} \lor \ldots \lor \ell_{i,k_i}$, where each $\ell_{i,j}$ is a literal, i.e. either a propositional variable $p_m$ or its negation $\neg p_m$. For example, the formula $(p \lor \neg q \lor r) \land (\neg p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ is in CNF.

Show that every formula is equivalent to a formula in conjunctive normal form.

2. (10 pts) Let $\mid$ be the propositional connective nand, i.e. $p \mid q$ is equivalent to $\neg(p \land q)$. Prove that the set $\{\mid\}$ is functionally complete.

3. (15 pts) Recall that a partial order on a set $X$ is a binary relation $<$ such that
   - if $x < y$ then $y \not< x$
   - if $x < y$ and $y < z$ then $x < z$.

A linear order is a partial order which also has the property that for any distinct $x, y \in X$, we have $x < y$ or $y < x$.

We say that a linear order $<'$ on $X$ extends a partial order $<$ on $X$ if whenever $x < y$ we have $x <' y$. Prove that if $X$ is a countable set, then every partial order on $X$ has a linear order which extends it. (Hint: prove the case where $X$ is a finite set first.)

4. (15 pts) (No collaboration) Let $S = \{\phi_1, \phi_2, \phi_3, \ldots\}$ be an infinite set of formulas. Assume for every valuation $v$ of the variables in the formulas of $S$, there is some $n$ (depending on $v$) such that $\phi_n$ is true for this valuation $v$. Prove that there is some fixed $m$ such that $\phi_1 \lor \phi_2 \lor \ldots \lor \phi_m$ is a tautology.