Homework 2: Due Tuesday April 14, 1pm

1. (10 pts) Say that a formula ϕ is in *conjunctive normal form* (CNF) if ϕ is of the form $\phi = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n$, where each ψ_i is of the form $\psi_i = \ell_{i,1} \vee \ldots \vee \ell_{i,k_i}$, where each $\ell_{i,j}$ is a *literal*, i.e. either a propositional variable p_m or its negation $\neg p_m$. For example, the formula $(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is in CNF.

   Show that every formula is equivalent to a formula in conjunctive normal form.

2. (10 pts) Let $|$ be the propositional connective *nand*, i.e. $p|q$ is equivalent to $\neg(p \wedge q)$. Prove that the set $\{|\}$ is functionally complete.

3. (15 pts) Recall that a *partial order* on a set X is a binary relation $<$ such that

   - if $x < y$ then $y \not< x$
   - if $x < y$ and $y < z$ then $x < z$.

   A *linear order* is a partial order which also has the property that for any distinct $x, y \in X$, we have $x < y$ or $y < x$.

   We say that a linear order $<'$ on X *extends* a partial order $<$ on X if whenever $x < y$ we have $x <' y$. Prove that if X is a countable set, then every partial order on X has a linear order which extends it. (Hint: prove the case where X is a finite set first.)

4. (15 pts) (No collaboration) Let $S = \{\phi_1, \phi_2, \phi_3, \ldots\}$ be an infinite set of formulas. Assume for every valuation v of the variables in the formulas of S, there is some n (depending on v) such that ϕ_n is true for this valuation v. Prove that there is some fixed m such that $\phi_1 \vee \phi_2 \vee \ldots \vee \phi_m$ is a tautology.