# Homework 1: Due Tuesday April 7, 1pm

1. (8 pts) Determine whether each of the following is a tautology, a contradiction or neither. Justify your answers.

   (a) $((p \to q) \to p) \to p$

   (b) $(q \wedge (p \to q)) \to p$

   (c) $p \wedge (p \leftrightarrow q) \wedge \neg q$

   (d) $((p \to q) \to r) \leftrightarrow (p \to (q \to r))$

2. (12 pts) Prove that all possible ways of adding parentheses to the expression "$p_1 \vee p_2 \ldots p_n$ " to create a formula yield equivalent formulas, which are true if and only if at least one of the $p_i$ is true. So for example $(p_1 \vee p_2) \vee (p_3 \vee p_4)$ is equivalent to $(p_1 \vee (p_2 \vee p3)) \vee p_4$ which are both true if and only if at least one of the $p_i$ is true. This justifies our occasional omission of parentheses when we write a disjunction of several propositions.

3. (10 pts) Recall that a set $S$ of formulas is called satisfiable if there is a single valuation which makes all the formulas in $S$ true. For each number $n = 2, 3, 4, \ldots$ find a set $S = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of $n$ formulas that is not satisfiable, but so that every proper nonempty subset $S'$ of $S$ is satisfiable.

4. (10 pts)

   (a) Prove that the set $\{\neg, \vee\}$ is functionally complete.

   (b) Prove that the set $\{\to, \vee\}$ is not functionally complete.

5. (15 pts) Let $\varphi$ be a formula containing only connectives from the set $\{\neg, \wedge, \vee\}$, and let $\varphi^*$ be the formula obtained from $\varphi$ by replacing each instance of $\wedge$ with $\vee$, each instance of $\vee$ with $\wedge$, and each of its propositional variables $p_i$ with $(\neg p_i)$. (For instance, if we have $\varphi = (p_1 \vee (\neg p_2)) \wedge p_3$, then we obtain $\varphi^* = ((\neg p_1) \wedge (\neg(\neg p_2))) \vee (\neg p_3))$.

   Prove using structural induction that $\neg\varphi$ is equivalent to $\varphi^*$.

   (Homework continued on the next page)

A set $S$ of formulas *logically implies* a formula $\phi$ if every valuation which satisfies $S$ also satisfies $\phi$.

6. (20 pts) (No collaboration) A set $S$ of formulas is said to be *independent* if for any formula $\varphi \in S$, $\varphi$ is not implied logically by the rest of the formulas in $S$. (So for example, the empty set $\emptyset$ is independent, and a set $S = \{\varphi\}$ consisting of a single formula is independent iff $\varphi$ is not a tautology).

   (a) Which of the following sets are independent? Justify your answers.

      i. $\{p \to q, q \to r, r \to q\}$

      ii. $\{p \vee q, p \to q, p \leftrightarrow q\}$

      iii. $\{p \to q, q \to r, p \to r\}$

   (b) Two sets of formulas $S$ and $R$ are called equivalent if $S$ logically implies every formula $\varphi \in R$ and $R$ logically implies every formula $\psi \in S$. Prove that for any *finite* set $S$ of formulas, there is a subset $S' \subseteq S$ which is independent and $S'$ is equivalent to $S$.