6c Lecture 11: May 5, 2015

9 Theories

Definition 9.1. Suppose that S is a set of sentences in a language L. We say that a structure with the language L *satisfies* T or *models* S and we write $M \vDash S$ if for every sentence $\phi \in S$ we have $M \vDash \phi$.

Definition 9.2. If S is a set of sentences in a language L, and ϕ is a sentence in L, then we say S logically implies ϕ and write $S \vDash \phi$ if for every model M of S, we have $M \vDash \phi$.

Definition 9.3. Let S be a set of sentences in a first-order language L. Then

$$\text{Con}(S) = \{\phi \mid S \vDash \phi, \phi \text{ a sentence}\}$$

Note that $\text{Con}(\text{Con}(S)) = \text{Con}(S)$. We also remark that this is somewhat nonstandard notation; Con is short for "consequences", so $\text{Con}(S)$ is the set of logical consequences of S. However, it is more common for Con to be short for "consistency" and refer to something else entirely. It won't come up in this class more than in passing, though.

Definition 9.4. A set of sentences T is called a *theory* if it is closed under logical consequences, i.e. if $\text{Con}(T) = T$.

Definition 9.5. If $\text{Con}(S) = T$, we say S is a set of axioms for T. Note that in this case if $M \vDash S$, then $M \vDash T$.

We spent most of this lecture giving examples of theories.

Example 1. Let $L = \{R\}$ with R a binary relation. Look at the following sentences.

1. $\forall x \forall y (xRy \rightarrow \neg(yRx))$
2. $\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow (xRz))$
3. $\forall x \forall y (xRy \vee x = y \vee yRx)$

Sentences 1 and 2 are axioms for the theory of partial orders, meaning that any model of 1 and 2 is called a partial order with respect to R. Adding in sentence 3 gives you the axioms for the theory of linear orders.

You could add in more axioms to get even more special linear orders.

4. $\exists x \exists y (x \neq y)$

5. $\forall x \forall y(xRy \to \exists z(xRz \wedge zRy))$

This axioms tell you that the linear order is dense. So $\langle \mathbb{Q}; < \rangle$ and $\langle \mathbb{R}; < \rangle$ model sentences 1-5 (with R interpreted as the usual $<$ relation), but $\langle \mathbb{N}; < \rangle$ does not. If you add in sentences saying that the order has no endpoints (exercise: write down such sentences), then you get axioms for the theory of dense linear orders without endpoints. This theory turns out to be very special for reasons that we don't have time to get into in this class. Take 116 to find out more.

Definition 9.6. A well-order is a linear order $\langle A; < \rangle$ with the property that every nonempty $S \subseteq A$ has a $<$-least element.

The best-known example of a well-order is $\langle \mathbb{N}; < \rangle$. It turns out that there is no first-order theory whose models are exactly the well-orders. We'll see why this is true in a few lectures.

Example 2. We discussed axioms for the theories of groups, abelian groups, commutative rings with identity, integral domains, fields, and fields of characteristic 0. These axioms all build on each other. We mentioned that it requires infinitely many sentences to axiomatize the theory of fields of characteristic 0. As with the other things we've mentioned so far, we'll see why soon. The reasons are all related!

Here is another source of examples of theories.

Definition 9.7. Let M be some L-structure. The *theory* of M is

$$\mathrm{Th}(M) = \{\phi \mid \phi \text{ is a sentence and } M \vDash \phi\}$$

The issue with using this to define a theory is that it is generally difficult to find axioms for $\mathrm{Th}(M)$, and working from axioms is the way we are able to do most of our reasoning about models. Sometimes we are lucky though.

Example 3. Let $L = \{0, 1, +, \cdot\}$ and $M = \langle \mathbb{R}; 0, 1, +, \cdot \rangle$. Then $\mathrm{Th}(M)$ is axiomatized by

- the field axioms
- $\forall x \exists y(x = y^2 \vee x + y^2 = 0)$
- $\forall x \forall y \exists z(x^2 + y^2 = z^2)$
- $\forall x(x^2 \neq -1)$ (here -1 is the additive inverse of 1, it's not in our language but it's definable so we use the shorthand)
- the infinitely many sentences

  $$\psi_n = \forall x_0 \ldots \forall x_n(x_n \neq 0 \to \exists y(x_n \cdot y^n + x_{n-1} \cdot y^{n-1} + \ldots + x_1 \cdot y + x_0 = 0)$$

  for n odd. In words, this says that any odd-degree polynomial has a root.

It is *not* obvious that these suffice to axiomatize M, and proving this is a bit beyond the scope of the course.

Example 4. The model $\langle \mathbb{C}; 0, 1, +, \cdot \rangle$ is axiomatized by the axioms for fields of characteristic 0 along with the sentences ψ_n for all n, not just the odd n.

Example 5. Consider $L = \{E\}$, where E is a binary relation. Then the following sentences axiomatize the theory of (undirected) graphs, where E is just the relation of having an edge between two vertices.

1. $\forall x(\neg xEx)$
2. $\forall x \forall y(xEy \to yEx)$

However, there is no first-order theory of connected graphs. Again, we will see why soon-ish.

Example 6. Let $L_{ar} = \{0, S, +, \cdot, <\}$, where S is a unary function and everything else has the form you'd expect (e.g. $+$ is a binary function). Let $\mathcal{N} = \langle \mathbb{N}; 0, S, +, \cdot; < \rangle$ where S is interpreted as the successor function, i.e. $S(x) = x + 1$. The following axioms are called the Peano axioms:

- $\forall x(S(x) \neq 0)$
- $\forall x \forall y(x \neq y \to S(x) \neq S(y))$
- $\forall x(x + 0 = x)$
- $\forall x \forall y(x + S(y) = S(x + y))$
- $\forall x(x \cdot 0 = 0)$
- $\forall x \forall y(x \cdot S(y) = x \cdot y + x)$
- the infinitely many sentences (one for each formula $\phi(x, y_1, \ldots, y_n)$) of the form

  $$\forall y_1 \ldots \forall y_n((\phi(0, y_1, \ldots, y_n) \wedge \forall x(\phi(x, y_1, \ldots, y_n) \to \phi(S(x), y_1, \ldots, y_n))) \to \forall x \phi(x, y_1, \ldots, y_n))$$

  These sentences are saying that you can do proofs by induction.

These axioms together seem to have more or less every known fact of number theory as a logical consequence. That said, we know that $\text{Con}(PA) \neq \text{Th}(\mathcal{N})$. In fact, one can show that there is no "reasonable" set of axioms for $\text{Th}(\mathcal{N})$. As you can imagine, this is a bit vexing.

The language of set theory is the language containing a single binary relation ϵ. Then ZFC is the theory containing the sentences:

1. $\exists x(x = x)$

2. $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$

3. $\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))]$

4. For each formula ϕ with free variables $x, z, w_1, \ldots, w_n$,

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \phi)$$

5. $\forall x \forall y \exists z (x \in z \wedge y \in z)$

6. $\forall f \exists a \forall y \forall x (x \in y \wedge y \in f \rightarrow x \in a)$

7. $\vdots$

Most all modern mathematics can be formalized and proved in ZFC. But we still find, as with PA, that Con(ZFC) can not be "reasonably" axiomatized.