6c Lecture 10: April 30, 2015

8 Logical equivalence, prenex normal form and games

8.1 Prenex normal form

Definition 8.1. If ϕ and ψ are first-order formulas in the language L with free variables $x_1, \ldots, x_n$, then we say ϕ and ψ are *equivalent*, and we write $\phi \equiv \psi$, if for every every structure M in the language L, and every n-tuple $(a_1, \ldots, a_n)$, we have $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_n \mapsto x_n]$ iff $M \vDash \psi[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]$.

For example, if ϕ is a formula containing the variable x and not containing the variable y, then if we replace each instance of x by y, we obtain an equivalent formula. (You can prove this formally by induction on formulas).

Another sources of equivalences in first-order logic is equivalences from propositional logic. For example, the propositional formulas $p \to q$ and $\neg q \to \neg p$ are equivalent, so $\phi = \exists x(R(x)) \to \exists y(S(y))$ and $\psi = \neg(\exists y(S(y))) \to \neg(\exists x(R(x)))$ are also equivalent. Indeed, if we take any two formulas of propositional logical logic and replace each variable p_i in these formulas with a first-order formula ψ_i, then the resulting formulas of first-order logic will also be equivalent.

We have some important equivalences to do with manipulating quantifiers:

Proposition 8.2. *Suppose ϕ is a formula. Then:*

- $\neg(\exists x(\phi)) \equiv \forall x(\neg\phi)$.
- $\neg(\forall y((\phi)) \equiv \exists x(\neg\phi)$.

Further, if ψ is a formula that does not have x as a free variable, then

- $(\forall x(\phi)) \wedge \psi \equiv \forall x(\phi \wedge \psi)$.
- $(\forall x(\phi)) \vee \psi \equiv \forall x(\phi \vee \psi)$.
- $(\exists x(\phi)) \wedge \psi \equiv \exists x(\phi \wedge \psi)$.
- $(\exists x(\phi)) \vee \psi \equiv \exists x(\phi \vee \psi)$.

Proof. In class. This follows directly from the definition of equivalence and the satisfaction relation ψ. Keep in mind that $M \vDash \psi[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n, x \mapsto b]$ iff $M \vDash \psi[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]$ since x is not free in ψ. □

Next, we define a useful normal form for first-order formulas, and show that every formula is equivalent to a formula in this normal form.

Definition 8.3. A first-order formula is said to be in *prenex normal form* if it is of the form $Q_1x_1Q_2x_2\dots Q_nx_n\theta$ where each Q_i is either the quantifier $\exists$ or $\forall$, and θ is a quantifier-free formula.

For example, $\exists x\forall y\exists z(R(x,y)\to(R(x,z)\vee\neg(R(y,z))))$ is in prenex normal form. The formula $\exists x(R(x)\to\exists y(R(x,y)\wedge\forall z(R(y,z)\to R(x,z))))$ is not in prenex normal form.

It turns out that along with technique of changing variable names, the equivalences given by Proposition 8.2 are all we need to transform any formula into an equivalent one in prenex normal form. We give an example:

$$
\begin{aligned}
&\neg\exists x(R(x)\wedge(\forall y(S(x,y))\vee\neg\forall y(T(x,y)))\\
&\qquad\equiv\neg\exists x(R(x)\wedge(\forall y(S(x,y))\vee\exists y(\neg T(x,y)))\\
&\qquad\equiv\neg\exists x(R(x)\wedge(\forall y(S(x,y))\vee\exists z(\neg T(x,z)))\\
&\qquad\equiv\neg\exists x(R(x)\wedge\forall y(S(x,y)\vee\exists z(\neg T(x,z)))\\
&\qquad\equiv\neg\exists x(R(x)\wedge\forall y\exists z(S(x,y)\vee(\neg T(x,z)))\\
&\qquad\equiv\neg\exists x\forall y\exists z(R(x)\wedge(S(x,y)\vee(\neg T(x,z)))\\
&\qquad\equiv\forall x\exists y\forall z[\neg(R(x)\wedge(S(x,y)\vee(\neg T(x,z)))]
\end{aligned}
$$

Theorem 8.4. *If ϕ is a first-order formula, then there is a first-order formula ϕ^* in prenex normal form with the same set of free variables such that ϕ is equivalent to ϕ^*.*

Proof. By induction on formula complexity. First, since the logical connectives $\wedge,\vee,\neg$ are functionally complete, it suffices to prove this for formulas using only these connectives.

For our base case, note that every quantifier-free formula is already in prenex normal form.

For our inductive case, suppose ϕ and ψ are formulas equivalent to $\phi^*=Q_1x_1\dots Q_nx_n(\theta)$ and $\psi^*=Q'_1y_1\dots Q'_ny_n(\xi)$ in PNF, where θ and ξ are quantifier-free. Then

($\exists$): $\exists x\phi\equiv\exists x\phi^*$ which is in PNF.

($\forall$): $\forall x\phi\equiv\forall x\phi^*$ which is in PNF.

($\neg$): $\neg\phi\equiv\neg\phi^*\equiv\neg Q_1x_1\dots Q_nx_n(\theta)\equiv\overline{Q_1}x_1\dots\overline{Q_n}x_n(\neg\theta)$ which is in PNF where $\overline{Q_i}$ is $\exists$ if Q_i is $\forall$, and $\overline{Q_i}$ is $\forall$ if Q_i is exist.

($\wedge$): By changing variables, we can assume assume the set of variables $x_1\dots x_n$ is disjoint from $y_1\dots y_m$. We can also assume that the variables $x_1,\dots,x_n$ are not free in ξ, and the variables $y_1,\dots,y_n$ are not free in θ. Then

$$\phi\wedge\psi\equiv\phi^*\wedge\psi^*\equiv Q_1x_1\dots Q_nx_nQ'_1y_1\dots Q'_ny_n(\theta\wedge\xi)$$

which is in PNF. The equivalence follows by repeatedly applying our previous proposition.

($\vee$): Again, assuming $x_1 \ldots x_n$ is disjoint from $y_1 \ldots y_m$, then $\phi \vee \psi \equiv \phi^* \vee \psi^* \equiv Q_1 x_1 \ldots Q_n x_n Q'_1 y_1 \ldots Q'_n y_n (\theta \vee \xi)$ which is in PNF.

□

8.2 Games and quantifiers

Suppose we have a formula $\phi = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots (R(x_1, \ldots, x_n))$ with alternating quantifiers, a structure M having universe A and an n-ary relation R. Then we can associate a game $G_{\phi,M}$ to this formula and structure, as follows. The game has two players $\exists$ and $\forall$, who alternate playing elements $a_1, a_2, \ldots, a_n$ of the universe of A, with $\exists$ going first:

$$\begin{array}{cc} \exists & \forall \\ a_1 & \\ & a_2 \\ a_3 & \\ & a_4 \\ \vdots & \end{array}$$

Once the game is finished, then $\exists$ wins if $R(a_1, \ldots, a_n)$ is true. Otherwise, $\forall$ wins.

For example, suppose $R(x_1, x_2, x_3)$ is the relation on $\mathbb{N}$ defined by $R(x_1, x_2, x_3)$ iff $x_3 + x_3 = x_2 \vee x_3 + x_3 + x_1 = x_2$, ϕ is the formula $\exists x_1 \forall x_2 \exists x_3 R(x_1, x_2, x_3)$, and we work in the model $\langle \mathbb{N}; + \rangle$. Then here is one play of the game $G_{\phi,\langle N;+\rangle}$, where the first move of $\exists$ is 1, $\forall$ makes the move 9, and $\exists$ finishes by playing 4:

$$\begin{array}{cc} \exists & \forall \\ 1 & \\ & 9 \\ 4 & \end{array}$$

Here $\exists$ wins this play of the game, since $4 + 4 = 9 \vee 4 + 4 + 1 = 9$ is true.

Recall that a *winning strategy* for a player is a way of making a move for this player on each turn of the game so that no matter what moves their opponent makes, once the game finished, this player wins. For example, $\exists$ has a winning strategy in our game above; they should play $a_1 = 1$ first, and then after $\forall$ plays a_2, then $\exists$ should play $a_3 = a_2/2$ if a_2 is even, and $a_3 = (a_2 - 1)/2$ if a_2 is odd.

We can now give a game-theoretic interpretation of when ϕ is true. The analysis we are about to do will also transfer easily to any formula in prenex normal form.

Theorem 8.5. *Suppose $\phi = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots (R(x_1, \ldots, x_n))$ is a sentence with alternating quantifiers in the language of a structure M. Then*

1. *$M \vDash \phi$ iff $\exists$ has a winning strategy in the game $G_{\phi,M}$.*
2. *$M \vDash \neg\phi$ iff $\forall$ has a winning strategy in the game $G_{\phi,M}$.*

Proof. Suppose $M \vDash \phi$. Then by definition there is some $a_1 \in A$ such that $M \vDash \forall x_2 \exists x_3 \ldots (R(x_1, \ldots, x_n))[x_1 \mapsto a_1]$. Have $\exists$ play a_1. By definition, no matter what $a_2 \in A$ $\forall$ plays, $M \vDash \exists x_3 \ldots (R(x_1, \ldots, x_n))[x_1 \mapsto a_1, x_2 \mapsto a_2]$. Continuing in this fashion, we find no matter what $\forall$ plays, there are $a_1, a_3, a_5, \ldots \in A$ that $\exists$ can play them and we arrive at $M \vDash R(a_1, \ldots, a_n)$.

Similar reasoning shows that $\forall$ has a winning strategy if $M \vDash \neg\phi$. The other direction of the above statements follows from the fact that if one player has a winning strategy, then the other does not. $\square$

From this we can also deduce a famous classical theorem of game theory:

Corollary 8.6 (Zermelo)**.** *Any two-player game of finite length and perfect information has a winning strategy for one of the two players.*