7 Definability and automorphisms

**Definition 7.1.** If $M$ is a structure with universe $A$, then we say that an element $a \in A$ is *(first-order)* definable in $M$ if there is a first-order formula $\phi$ with one free variable $x$ such that $a$ is the unique element of $A$ such that $M \models \phi(x)$ is true under the assignment $x \mapsto a$.

**Example 7.2.** 5 is definable in the structure $\langle \mathbb{N}; 0, 1, + \rangle$, via the formula $x = 1 + 1 + 1 + 1 + 1$.

**Example 7.3.** $\sqrt{2}$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. Since $\sqrt{2}$ is the only positive solution of $x^2 = 2$, it is defined by the formula

$$(x \cdot x = 2) \land \exists y (y \cdot y = x)$$

**Example 7.4.** $\pi$ is not definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. We do not give a proof here, but it is an easy consequence of the Tarski-Seidenberg theorem which we will discuss later in class, and Lindemann’s theorem that $\pi$ is a transcendental number.

**Definition 7.5.** If $M$ is a structure with universe $A$, then we say that a relation $R$ on $A$ is *(first-order)* definable in $M$ if there is first-order formula $\phi$ with $n$ free variables $x_1, \ldots, x_n$ such that for all $n$-tuples $(a_1, \ldots, a_n)$, we have

$$R(a_1, \ldots, a_n) \leftrightarrow M \models \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$$

**Example 7.6.** The relation $<$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$ since $x < y$ iff $x \neq y \land \exists z (x + z = y)$.

Recall that we can identify 1-ary relations on a set $A$ with subsets of $A$. Hence, we will often say that a set $X \subseteq A$ is definable if it is definable as a 1-ary relation.

**Example 7.7.** The set $\mathbb{N}$ is definable in the structure $\langle \mathbb{Z}; 0, 1, +, \cdot \rangle$. We can see this via Lagrange’s four square theorem. Since every nonnegative integer $n$ can be written as a sum of four integer squares $n = m_1^2 + m_2^2 + m_3^2 + m_4^2$, we have that $\mathbb{N}$ is definable via the formula:

$$\exists m_1 \exists m_2 \exists m_3 \exists m_4 (x = m_1 \cdot m_1 + m_2 \cdot m_2 + m_3 \cdot m_3 + m_4 \cdot m_4)$$

Finally, we similarly have a notation of definability for functions:
Definition 7.8. If $M$ is a structure with universe $A$, then we say that a $n$-ary function $f$ on $A$ is \textit{(first-order) definable} in $M$ if there is first-order formula $\phi$ with $n+1$ free variables $x_1, \ldots, x_n, x_{n+1}$ such that for all $(n+1)$-tuples $(a_1, \ldots, a_n, a_{n+1})$, we have

$$f(a_1, \ldots, a_n) = a_{n+1} \iff M \models \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$$

Example 7.9. The function $f(x) = \sqrt[3]{x}$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$, using the formula $x_1 \cdot x_1 \cdot x_1 = x_2$.

7.1 The automorphism method

Definition 7.10. Suppose $M = \langle A; f^M_1, \ldots, f^M_i, \ldots, R^M_j \rangle$ and $N = \langle B; f^N_1, \ldots, f^N_i, \ldots, R^N_j \rangle$ are structures with the same signature. Then an \textit{isomorphism} from $M$ to $N$ is a bijection (a 1-1 and onto function) $\pi: A \to B$ such that for every $n$-ary function $f_i$, and every $n$-tuple $(a_1, \ldots, a_n) \in A^n$,

$$\pi(f^M_i(a_1, \ldots, a_m)) = f^N_i(\pi(a_1), \ldots, \pi(a_m)),$$

and for every $n$-ary relation $R_i$, and every $n$-tuple $(a_1, \ldots, a_n) \in A^n$,

$$\pi(R^M_i(a_1, \ldots, a_m)) \iff R^N_i(\pi(a_1), \ldots, \pi(a_m)).$$

If there is an isomorphism from $M$ to $N$, then we say $M$ and $N$ are \textit{isomorphic}.

We give a picture illustrating the equation

$$\pi(f^M_i(a_1, \ldots, a_m)) = f^N_i(\pi(a_1), \ldots, \pi(a_m)).$$

If $M$ is isomorphic to $N$, then you should think of $M$ and $N$ as being the same structure, just with the universe of $N$ being a “relabeled” version of the universe of $M$ via the function $\pi$. 

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Example 7.11. Consider the graphs $G_1$ and $G_2$ on the set of vertices $\{1, 2, 3, 4\}$ and $\{a, b, c, d\}$ respectively, and having an edge relations $E^{G_1}$ and $E^{G_2}$ as follows: $G_1 = \langle 1, 2, 3, 4, \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (1, 4), (4, 1)\} \rangle$, and $G_2 = \langle a, b, c, d, \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (a, d), (d, a)\} \rangle$. Then the function $\pi$ where $\pi(1) = a$, $\pi(2) = b$, $\pi(3) = c$, and $\pi(4) = d$, is an isomorphism from $G_1$ to $G_2$, since we can check that for every $(a_1, a_2)$ in the universe of $G_1$, we have 

$$E_1^G(a_1, a_2) \iff E_2^G(\pi(a_1), \pi(a_2)).$$

We draw a picture below:

![Graphs G1 and G2](image)

Example 7.12. Consider the structures $\langle \mathbb{R}^+; \cdot \rangle$ and $\langle \mathbb{R}; + \rangle$, where $\mathbb{R}^+$ is the set of positive integers. The function $\pi(x) = \log x$ is an isomorphism from $\langle \mathbb{R}^+; \cdot \rangle$ to $\langle \mathbb{R}; + \rangle$. To check this, for the single functions in these two structures, we must show that for every $(a_1, a_2) \in (\mathbb{R}^+)^2$, we have:

$$\pi(a_1 \cdot a_2) = \pi(a_1) + \pi(a_2)$$

which is equivalent to

$$\log(a_1 \cdot a_2) = \log(a_1) + \log(a_2)$$

which is a law of logarithms.

**Theorem 7.13.** Suppose $\pi$ is an isomorphism between structures $M$ and $N$ having the same language, $\phi$ is a formula in this language having free variables $x_1, \ldots, x_n$, and $(a_1, \ldots, a_n)$ is an $n$-tuple of elements of the universe of $M$. Then $M \models \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$ iff $N \models \phi[x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)]$.

**Proof.** In class, by induction on formulas. Remember that you first start by showing that terms work the way you expect, meaning that

$$\pi(tM[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]) = tN([x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)].$$
Then work up to atomic formulas, and then to formulas. This comes down to definitions (either of satisfaction or of an isomorphism) at just about all points; every so often you need to use the fact that \( \pi \) is a bijection.

For example, consider our isomorphism above between the graphs \( G_1 \) and \( G_2 \). Then the formula \( \phi = \exists y \exists z (xEy \land xEZ \land yEZ) \) is true in \( G_1 \) when \( x \mapsto 1 \) and therefore \( \phi \) is also true in \( G_2 \) when \( x \mapsto a \), since \( \pi(1) = a \). (Similarly \( \phi \) is false in \( G_1 \) when we assign \( x \mapsto 4 \) and \( \phi \) is false in \( G_2 \) when we assign \( x \mapsto d \).)

**Definition 7.14.** An automorphism of a structure \( M \) is an isomorphism from \( M \) to \( M \). For every structure, the identity function \( \pi(x) = x \) is an automorphism of \( M \). This automorphism is called the trivial automorphism, and an automorphism is called nontrivial if it is not equal to the identity automorphism.

A corollary of Theorem 7.13 gives a very useful technique for proving functions and relations are not first-order definable.

**Corollary 7.15.** If \( \pi \) is an automorphism of \( M \), then for every formula \( \phi \) with \( n \) free variables \( x_1, \ldots, x_n \) and every \( n \)-tuple \( a_1, \ldots, a_n \) in the universe of \( M \),

\[
M \vDash \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n] \iff M \vDash \phi[x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)]
\]

**Example 7.16.** The function \( \pi(a) = a^3 \) is an automorphism of the structure \( \langle \mathbb{R}; 0, 1, \cdot \rangle \), since \( \pi(0) = 0, \pi(1) = 1, \) and for every \( a, b \in \mathbb{R} \)

\[
\pi(a \cdot b) = \pi(a) \cdot \pi(b)
\]

is true, since

\[
(a \cdot b)^3 = a^3 \cdot b^3.
\]

Note that \( \pi(x) = x^2 \) is not an automorphism of \( \langle \mathbb{R}; 0, 1, \cdot \rangle \) since \( \pi \) is not a bijection.

**Example 7.17.** \( \mathbb{N} \) is not definable in \( \langle \mathbb{R}; 0, 1, \cdot \rangle \). We can prove this by using Corollary 7.15 and the automorphism \( \pi(x) = x^3 \) given above. By way of contradiction, if \( \mathbb{N} \) was definable, then there would be a formula \( \phi \) such that \( \langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto a] \) iff \( a \in \mathbb{N} \). So \( \langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto \sqrt{2}] \) would have to be false, but this is true iff \( \langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto 2] \) by Corollary 7.15. However, \( \langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto 2] \) must be true since \( \phi \) defines \( \mathbb{N} \). Contradiction!