6c Lecture 9: April 28, 2015

7 Definability and automorphisms

Definition 7.1. If M is a structure with universe A, then we say that an element $a \in A$ is *(first-order) definable* in M if there is a first-order formula ϕ with one free variable x such that a is the unique element of A such that $M \vDash \phi(x)$ is true under the assignment $x \mapsto a$

Example 7.2. 5 is definable in the structure $\langle \mathbb{N}; 0, 1, + \rangle$, via the formula $x = 1 + 1 + 1 + 1 + 1$.

Example 7.3. $\sqrt{2}$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. Since since $\sqrt{2}$ is the only positive solution of $x^2 = 2$, it is defined by the formula

$$(x \cdot x = 2) \wedge \exists y(y \cdot y = x)$$

Example 7.4. π is not definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. We do not give a proof here, but it is an easy consequence of the Tarski-Seidenberg theorem which we will discuss later in class, and Lindemann's theorem that π is a transcendental number.

Definition 7.5. If M is a structure with universe A, then we say that a relation R on A is *(first-order) definable* in M if there is first-order formula ϕ with n free variables $x_1, \ldots, x_n$ such that for all n-tuples $(a_1, \ldots, a_n)$, we have

$$R(a_1, \ldots, a_n) \leftrightarrow M \vDash \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$$

Example 7.6. The relation $<$ is definable in the structure $\langle \mathbb{R}; 0, 1, + \cdot \rangle$ since $x < y$ iff $x \neq y \wedge \exists z x + z \cdot z = y$

Recall that we can identify 1-ary relations on a set A with subsets of A. Hence, we will often say that a set $X \subseteq A$ is definable if it is definable as a 1-ary relation.

Example 7.7. The set $\mathbb{N}$ is definable in the structure $\langle \mathbb{Z}; 0, 1, +, \cdot \rangle$. We can see this via Lagrange's four square theorem. Since every nonnegative integer n can be written as a sum of four integer squares $n = m_1^2 + m_2^2 + m_3^2 + m_4^2$, we have that $\mathbb{N}$ is definable via the formula:

$$\exists m_1 \exists m_2 \exists m_3 \exists m_4 (x = m_1 \cdot m_1 + m_2 \cdot m_2 + m_3 \cdot m_3 + m_4 \cdot m_4)$$

Finally, we similarly have a notation of definability for functions:

Definition 7.8. If M is a structure with universe A, then we say that a n-ary function f on A is *(first-order) definable* in M if there is first-order formula ϕ with $n+1$ free variables $x_1, \ldots, x_n, x_{n+1}$ such that for all $(n+1)$-tuples $(a_1, \ldots, a_n, a_{n+1})$, we have

$$f(a_1, \ldots, a_n) = a_{n+1} \leftrightarrow M \vDash \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$$

Example 7.9. The function $f(x) = \sqrt[3]{x}$ is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$, using the formula $x_1 \cdot x_1 \cdot x_1 = x_2$.

7.1 The automorphism method

Definition 7.10. Suppose $M = \langle A; f_1^M, \ldots, f_i^M; R_1^M, \ldots, R_j^M \rangle$ and $N = \langle B; f_1^N, \ldots, f_i^N; R_1^N, \ldots R_j^N \rangle$ are structures with the same signature. Then an *isomorphism* from M to N is a bijection (a 1-1 and onto function) $\pi \colon A \to B$ such that for every $n-ary$ function f_i, and every n-tuple $(a_1, \ldots a_n) \in A^n$,

$$\pi(f_i^M(a_1, \ldots, a_m)) = f_i^N(\pi(a_1), \ldots, \pi(a_m)),$$

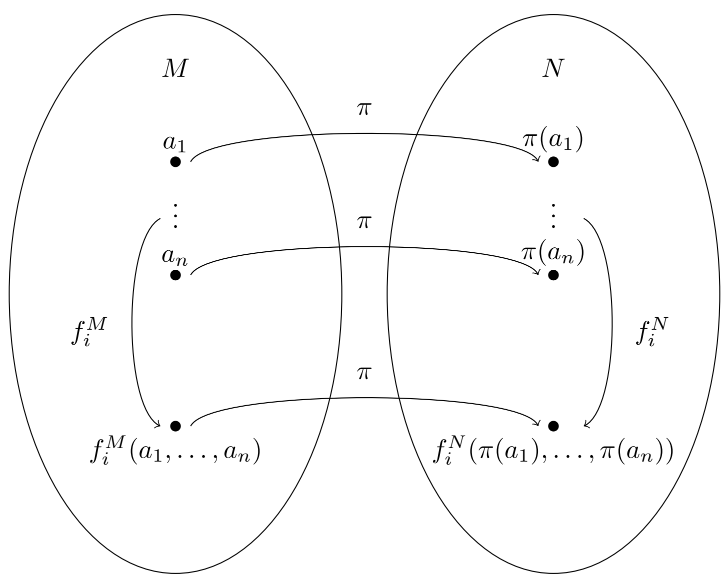
and for every n-ary relation R_i, and every n-tuple $(a_1, \ldots a_n) \in A^n$,

$$\pi(R_i^M(a_1, \ldots, a_m)) \leftrightarrow R_i^N(\pi(a_1), \ldots, \pi(a_m)).$$

If there is an isomorphism from M to N, then we say M and N are *isomorphic.*

We give a picture illustrating the equation

$$\pi(f_i^M(a_1, \ldots, a_m)) = f_i^N(\pi(a_1), \ldots, \pi(a_m)).$$

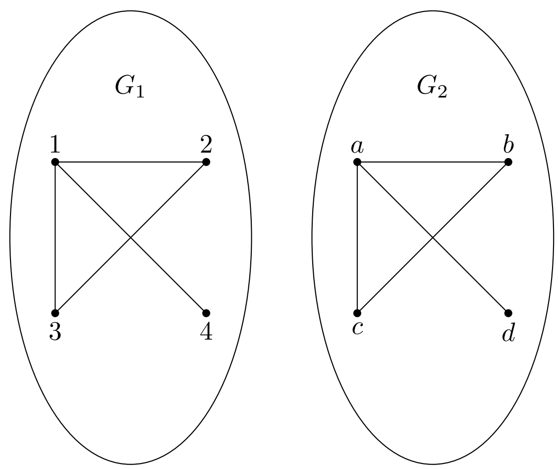


If M is isomorphic to N, then you should think of M and N as being the same structure, just with the universe of N being a "relabeled" version of the universe of M via the function π.

Example 7.11. Consider the graphs G_1 and G_2 on the set of vertices $\{1, 2, 3, 4\}$ and $\{a, b, c, d\}$ respectively, and having an edge relations E^{G_1} and E^{G_2} as follows: $G_1 = \langle 1, 2, 3, 4, \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (1, 4), (4, 1)\}\rangle$, and $G_2 = \langle a, b, c, d, \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (a, d), (d, a)\}\rangle$. Then the function π where $\pi(1) = a$, $\pi(2) = b$, $\pi(3) = c$, and $\pi(4) = d$, is an isomorphism from G_1 to G_2, since we can check that for every (a_1, a_2) in the universe of G_1, we have

$$E_1^G(a_1, a_2) \leftrightarrow E_2^G(\pi(a_1), \pi(a_2)).$$

We draw a picture below:



Example 7.12. Consider the structures $\langle \mathbb{R}^+; \cdot\rangle$ and $\langle R; +\rangle$, where $\mathbb{R}^+$ is the set of positive integers. The the function $\pi(x) = \log x$ is an isomorphism from $\langle \mathbb{R}^+; \cdot\rangle$ to $\langle \mathbb{R}; +\rangle$. To check this, for the single functions in these two structures, we mus show that for every $(a_1, a_2) \in (\mathbb{R}^+)^2$, we have:

$$\pi(a_1 \cdot a_2) = \pi(a_1) + \pi(a_2)$$

which is equivalent to

$$\log(a_1 \cdot a_2) = \log(a_1) + \log(a_2)$$

which is a law of logarithms.

Theorem 7.13. *Suppose π is an isomorphism between structures M and N having the same language, ϕ is a formula in this language having free variables $x_1, \ldots, x_n$, and $(a_1, \ldots, a_n)$ is an n-tuple of elements of the universe of M. Then $M \vDash \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]$ iff $N \vDash \phi[x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)]$.*

Proof. In class, by induction on formulas. Remember that you first start by showing that terms work the way you expect, meaning that

$$\pi(t^M[x_1 \mapsto a_1 \ldots x_n \mapsto a_n]) = t^N([x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)].$$

Then work up to atomic formulas, and then to formulas. This comes down to definitions (either of satisfaction or of an isomorphism) at just about all points; every so often you need to use the fact that π is a bijection. $\square$

For example, consider our isomorphism above beween the graphs G_1 and G_2. Then the formula $\phi = \exists y \exists z (xEy \wedge xEz \wedge yEz)$ is true in G_1 when $x \mapsto 1$ and therefore ϕ is also true in G_2 when $x \mapsto a$, since $\pi(1) = a$. (Similarly ϕ is false in G_1 when we assign $x \mapsto 4$ and ϕ is false in G_2 when we assign $x \mapsto d$.)

Definition 7.14. An automorphism of a structure M is an isomorphism from M to M. For every structure, the identity function $\pi(x) = x$ is an automorphism of M. This automorphism is called the *trivial automorphism*, and an automorphism is called *nontrivial* if it is not equal to the identity automorphism.

A corollary of Theorem 7.13 gives a very useful technique for proving functions and relations are not first-order definable.

Corollary 7.15. *If π is an automorphism of M, then for every formula ϕ with n free variables $x_1, \ldots, x_n$ and every n-tuple $a_1, \ldots, a_n$ in the universe of M,*

$$M \vDash \phi[x_1 \mapsto a_1 \ldots x_n \mapsto a_n] \leftrightarrow M \vDash \phi[x_1 \mapsto \pi(a_1) \ldots x_n \mapsto \pi(a_n)]$$

Example 7.16. The function $\pi(a) = a^3$ is an automorphism of the structure $\langle \mathbb{R}; 0, 1, \cdot \rangle$, since $\pi(0) = 0$, $\pi(1) = 1$, and for every $a, b \in \mathbb{R}$

$$\pi(a \cdot b) = \pi(a) \cdot \pi(b)$$

is true, since

$$(a \cdot b)^3 = a^3 \cdot b^3.$$

Note that $\pi(x) = x^2$ is not an automorphism of $\langle \mathbb{R}; 0, 1, \cdot \rangle$ since π is not a bijection.

Example 7.17. $\mathbb{N}$ is not definable in $\langle \mathbb{R}; 0, 1, \cdot \rangle$. We can prove this by using Corollary 7.15 and the automorphism $\pi(x) = x^3$ given above. By way of contradiction, if $\mathbb{N}$ was definable, then there would be a formula ϕ such that $\langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto a]$ iff $a \in \mathbb{N}$. So $\langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto \sqrt[3]{2}]$ would have to be false, but this is true iff $\langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto 2]$ by Corollary 7.15. However, $\langle \mathbb{R}; 0, 1, \cdot \rangle \vDash \phi[x \mapsto 2]$ must be true since ϕ defines $\mathbb{N}$. Contradiction!