6c Lecture 8: April 24, 2014

6 First order logic

Our next topic is first-order logic, which is a more complex system than propositional logic that incorporates functions, relations and quantifiers. Indeed, all essentially all theorems of modern mathematics can be formalized and proved in first-order logic.

6.1 Structures

We begin by recalling the definition of relations and functions.

Definition 6.1. A *n-ary* relation R on a set A is a subset of the n-tuples A^n of A. We write $R(a_1, \ldots, a_n)$ to indicate that the n-tuple $(a_1, \ldots, a_n)$ is in R.

A n-ary function f on a set A is a rule assigning each n-tuple $(a_1, \ldots, a_n)$ a unique value $f(a_1, \ldots, a_n) \in A$.

Example 6.2. Some relations on $\mathbb{R}$:

1. $L = \{(x, y) \in \mathbb{R}^2 : x < y\}$ is a 2-ary (or binary) relation on $\mathbb{R}$.
2. $E = \{(x, y) \in \mathbb{R}^2 : x = y\}$ is a binary relation on $\mathbb{R}$.
3. $Q = \{x \in \mathbb{R} : x \text{ is rational}\}$ is a 1-ary (or unary) relation on $\mathbb{R}$.
4. For each natural number n, the $(n+1)$-ary relation $R_n = \{(a_0, a_1, \ldots, a_n) :$ the polynomial $a_n x^n + \ldots + a_1 x + a_0 = x$ has n distinct real roots$\}$.

Some relations on the set A of binary strings:

1. $P = \{(s, t) : s \text{ is a prefix of } t\}$
2. $S = \{(s, t) : s \text{ is a substring of } t\}$

Note that we can identify 1-ary relations on A with subsets of A in the obvious way.

Example 6.3. Some examples of functions on $\mathbb{R}$:

1. $s(x) = x^2$ is a unary function
2. $a(x) = x + y$ is a binary function
3. $f_n(x_1, \ldots, x_n) = \sqrt{x_1^2 + \ldots + x_n^2}$.

Some examples of functions on the set A of binary strings:

1. The binary function $C(s, t)$ equal to the concatenation of s and t.
2. The ternary function $R(r, s, t)$ equal to the string obtained by replacing the first instance of r in t with the string s.

Sometimes when R is a 2-ary relation we will write $x\ R\ y$ instead of $R(x, y)$. For example, we generally write $x < y$ instead of $<(x, y)$ for the relation $<$.

Note that a 0-ary function takes no input, and outputs a single value in A. A more common name for 0-ary functions is *constants*. For example, the constant value π is a 0-ary function on $\mathbb{R}$.

0-ary relation are sometimes also considered. There are two such relations, the relation which is always true, and the relation which is always false. These are denoted $\top$ and $\perp$ respectively.

Definition 6.4. A *structure* consists of a set A called the *universe* of the structure, together with a collection of functions $f_1, f_2, \dots$ and relations $R_1, R_2, \dots$ on A. We write $\langle A, f_1, f_2, \dots, R_1, R_2, \dots \rangle$ to note this structure.

We give some examples:

Example 6.5.

1. Let A be the set of all binary strings. Then we can consider the structure $\langle A; C, R; P, S \rangle$ whose universe is A, and which has the functions C and R and relations P and S defined above.
2. The structure $\langle \mathbb{N}; 0, 1, +, \cdot; < \rangle$ whose universe is the natural numbers and which has the constants 0 and 1, the functions $+$ and $\cdot$, and the relation $<$.
3. The structure $\langle \mathbb{R}; 0, 1, \pi, +, \cdot, \sin, \cos; < \rangle$ consisting of the real numbers with the constants 0, 1, and π, the functions of addition, multiplication, sin and cos, and the relation $<$.

Note that for example, $\langle \mathbb{R}; \tan \rangle$ is *not* a structure since the tan is not a function on $\mathbb{R}$; its domain does not include $\pi/2$, for example. Similarly, we cannot have a function for subtraction on a structure whose universe is $\mathbb{N}$, since $1 - 2$ is not an element of $\mathbb{N}$.

Every structure has a *language* associated to it, which is just the information of what symbols we use for the functions and relations of language, and what their airities are. For example, the language of the the structure $\langle A; C, R; P, S \rangle$ has 2 binary functions C and R, a binary relation P and a ternary relation S.

When we have several structures that use the same language, and we want to emphasize what model a particular function or relation comes from, we will use superscripts to indicate this. For example, the language with 2 binary relations $+$ and $\cdot$ is the language of both the structure $\mathcal{R} = \langle \mathbb{R}; +, \cdot \rangle$ and the structure $\langle \mathbb{N}; +, \cdot \rangle$. It is also the language of the the structure $\mathcal{Z}_2 = \langle \{0, 1\}; +, \cdot \rangle$ of the integers modulo 2, where we define $\cdot$ as usual, but addition as $0 +^{\mathcal{Z}_2} 0 = 0$, $0 +^{\mathcal{Z}_2} 1 = 1$, $1 +^{\mathcal{Z}_2} 0 = 1$, and $1 +^{\mathcal{Z}_2} 1 = 0$.

Formally we have the following definition:

Definition 6.6. A language is a set of symbols representing a collection of functions and relations, and information about their arities (i.e. the n for which each the function or relation is n-ary).

6.2 The satisfaction relation

Now we will give the longest and most pedantic sequence of definitions we'll encounter this whole quarter. Intuitively, what we want to do is to define what first-order formulas are (formulas we can build using variables, quantification over variables, functions and relations) and what it means for a formula to be true in a structure.

For example, we'll see that $\phi = \forall x \forall y \exists z (x+z = y)$ and $\psi = \forall y[(x+y) \geq x \cdot y]$ are formulas, and ϕ is true in the structure $\langle \mathbb{R}; + \rangle$ and false in the structure $\langle \mathbb{N}; + \rangle$ while in the structure $\langle \mathbb{N}; +, \cdot \rangle$, ψ is true under the assignment $x \mapsto 1$ and ψ is false under the assignment $x \mapsto 2$.

The recursive definitions we will give will often be used in inductive proofs we give.

We'll start by defining terms. Terms are expressions that represent an element of the universe of a structure. They can be single variables (for which we will usually use the letters x, y, z, w or $x_1, x_2, \ldots$), or combinations of variables made using function. For example, for the structure $\langle \mathbb{R}; 0, 1; +, \cdot \rangle$, we have that $(1+x) \cdot y + 1$ is a term whose value under the assignment $[x \mapsto 3, y \mapsto 1/4]$ is 2.

Definition 6.7. Fix a language L and a structure M. We give an inductive definition of what the terms of L are, what the variables of a term are, and if t is a term having variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ are elements of the universe of M, what the value of t under the assignment $[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]$ is, which we write $t[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]$.

Our base case is that any variable x is a term, and the value of the term consisting of a single variable x under any the assignment $[x \mapsto a]$ is a.

Now inductively, if $t_1, \ldots t_n$ are terms and f is an n-ary function in the language L, then $f(t_1, \ldots, t_n)$ is a term. Its set of variables is equal to the union of the variables appearing in $t_1, \ldots, t_n$, and if these variables are $x_1, \ldots, x_m$, then the value of the term $f(t_1, \ldots, t_n)[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ in M is equal to $f^M(t_1[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m], \ldots, t_n[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m])$.

Note that a 0-ary function (i.e. a constant) by itself is also a term.

Formally, each term in a language L having n variables defines an n-ary function on each structure for this language.

For example, lets consider again set of binary strings A, and lets work in structure $\langle A; 0, 1, C, S \rangle$ which has constants consisting of the strings "0" and "1", and the functions $C(s,t)$ of concatenation and the ternary function $R(r,s,t)$ equal to the string obtained by replacing the first instance of r in t with the string s. Then 0 is a term (just the constant 0). $C(0,1)$ is a term whose value is 01 in the structure. $R(C(0,1), C(1,0), x)$ is a term whose value under the assignment $[x \mapsto 000110]$ is 001010.

Next, we will give a definition of what first order formulas are; expressions such as:

$$\forall y((\exists x(x \cdot x + y = 0) \wedge \exists x(x \cdot x = y)) \to y = 0).$$

The quantifier $\exists x$ should be thought of as saying "there exists an x", and the quantifier $\forall x$ should be thought of as saying "for all x". Thus, for example, the above formula roughly says for every y, if there is an x such that $x^2 + y = 0$, and there is an x such that $x^2 = y$, then $y = 0$.

As usual, we define formulas recursively starting with terms combined using relations, and then adding logical connectives and quantifiers. We'll also define the satisfaction relation $\vDash$ for which given a model M indicates what formulas are true in M under what variable assignments.

Definition 6.8. Fix a language L and a structure M. We inductively define the class of first order formulas in the language L, the free variables in a formula, and and the satisfaction relation $\vDash$.

An *atomic formula* is a formula of one of the following two forms:

1. $R(t_1, \ldots, t_n)$, whenever where R is an n-ary relation of L and $t_1, \ldots, t_n$ are terms in L. If $x_1, \ldots, x_m$ are the variables occurring in the terms $t_1, \ldots, t_n$, then these variables are all said to be free in $R(t_1, \ldots, t_n)$, and if $a_1, \ldots, a_m$ is an m-tuple of elements in the universe of M, then $M \vDash R(t_1, \ldots, t_n)[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ if and only if in the structure M, $R^M(t_1[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m], \ldots, t_n[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m])$ is true.

2. $t_1 = t_2$, whenever t_1, t_2 are terms in L. If $x_1, \ldots, x_m$ are the variables occurring in the terms $t_1, \ldots, t_n$, then these variables are all said to be free in $t_1 = t_1$, and if $a_1, \ldots, a_m$ is an m-tuple of elements in the universe of M, then $M \vDash t_1 = t_2[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ if and only if in M $t_1[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m] = t_2[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$.

Now we obtain more complicated formulas by combining them with logical connectives and quantifiers. If ϕ and ψ are formulas, then

1. $\neg\phi$ is a formula, whose set of free variables is the same as ϕ. If these variables are $x_1, \ldots, x_m$, then under any assignment, $[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$, we have $M \vDash \neg\phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ iff its not the case that $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$.

2. $\phi \wedge \psi$ is a formula, whose set of free variables is the union of those of ϕ and ψ. If these variables are $x_1, \ldots, x_m$, then under any assignment, $[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$, we have $M \vDash \phi \wedge \psi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ iff $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ and $M \vDash \psi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$.

3. $\phi \vee \psi$, $\phi \to \psi$ and $\phi \leftrightarrow \psi$ are also formulas, and we define the satisfaction relation for them similarly.

If ϕ is a formula and y is a variable, then

1. $\exists y(\phi)$ is a formula, whose free variables are those of ϕ except y if it is a free variable in ϕ. We say that the variable y is *bound* by the quantifier $\exists y$, and that every instance of the variable y in ϕ is in the *scope* of the quantifier $\exists y$. If the free variables of $\exists y \phi$ are $x_1, \ldots, x_m$, then under the assignment $[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ we define $M \vDash \exists y(\phi)[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ iff there is an element b of the universe of M so that $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m, y \mapsto b]$.

2. $\forall y(\phi)$ is a formula, whose free variables are those of ϕ except y if it is a free variable in ϕ. We say that the variable y is *bound* by the quantifier $\forall y$, and that every instance of the variable y in ϕ is in the *scope* of the quantifier $\forall y$. If the free variables of $\forall y \phi$ are $x_1, \ldots, x_m$, then under the assignment $[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ we define $M \vDash \forall y(\phi)[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ iff for every element b of the universe of M, $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m, y \mapsto b]$.

Note that if ϕ is a formula whose free variables are included in $x_1, \ldots, x_n$, and $[x_1 \mapsto a_1, \ldots, x_m \mapsto a_m]$ is an assignment of values from the universe of M to variables $x_1, \ldots, x_m$, then we define $M \vDash \phi[x_1 \mapsto a_1, \ldots, x_m \mapsto a_1]$ to be the truth value of ϕ in M under the assignment which removes all variables that are not free in ϕ.

Note that we have included equality $=$ as a basic object in first order logic; because of the fundamental role that it plays in mathematics, it is difficult to to anything interesting without it.

Definition 6.9. A sentence ϕ is a formula which does not have any free variables

Sentences are important because they have truth values without specifying values for any of their variables.