5 Soundness and completeness of our Hilbert-style proof system

Our goal in this section is to prove the soundness and completeness of our Hilbert-style proof system. The easy half of this pair of theorems is soundness.

**Theorem 5.1.** Suppose $S = \{\phi_1, \phi_2, \ldots\}$ is a set of formulas, and $S \vdash \psi$. Then $S$ implies $\psi$.

**Proof.** By induction on the height of proofs. The guts of this proof is just checking that each type of step we use in our proofs is logically valid.

Base case: Proofs of height 0. Such a proof is just a statement of a logical axiom or a formula in $S$. If $\psi$ is a logical axiom, or $\psi = \phi_i$ for some formula $\phi_i$ in $S$, then clearly $S$ implies $\psi$.

Inductive case: Suppose that proofs of height $n$ are sound, and we have a proof that $S \vdash \psi$ of height $n + 1$, whose last step must be an instance of modus ponens concluding $\psi$ from the formulas $\theta$ and $\theta \rightarrow \psi$.

Now since there are proofs of $\theta$ and $\theta \rightarrow \psi$ using proofs of height $\leq n$, by our induction hypothesis, $S$ implies $\theta$ and $S$ implies $\theta \rightarrow \psi$. But then any valuation making $S$ true makes $\theta$ true and $\theta \rightarrow \psi$ true, and hence also makes $\psi$ true. Thus, $S$ implies $\psi$.

**Remark 5.3.** It follows from the definition of our proof system that if $S \vdash A \rightarrow B$, then $S \cup \{A\} \vdash B$. So the Deduction Theorem tells us that these two things are equivalent.

**Theorem 5.2** (The Deduction Theorem). Suppose that $S$ is a set of formulas and $A, B$ are formulas. If $S \cup \{A\} \vdash B$, then $S \vdash A \rightarrow B$.

**Proof.** This is an induction on the height of the proof of $B$. Suppose that $S \cup \{A\} \vdash B$ with a proof of height 1. There are three things that could be happening.
1. $B$ is a logical axiom. In this case $S \vdash B$. Also, since $B \rightarrow (A \rightarrow B)$ is an example of logical axiom 1, $S \vdash B \rightarrow (A \rightarrow B)$. Applying modus ponens, we find that $S \vdash A \rightarrow B$.

2. $B \in S$. Then $S \vdash B$ and the same argument as above works.

3. $B = A$. Then since $S \vdash A \rightarrow A$ by our example from last time, we find that $S \vdash A \rightarrow B$.

Now suppose inductively that $B$ follows from two statements by modus ponens, i.e., there is some formula $\psi$ such that $S \cup \{A\} \vdash \psi$ and $S \cup \{A\} \vdash \psi \rightarrow B$. By our inductive assumption, $S \vdash A \rightarrow \psi$ and $S \vdash A \rightarrow (\psi \rightarrow B)$. Using logical axiom 2, we find that

$$(A \rightarrow (\psi \rightarrow B)) \rightarrow ((A \rightarrow \psi) \rightarrow (A \rightarrow B))$$

is a logical axiom. Applying modus ponens twice, we find that $S \vdash A \rightarrow B$, as desired.

Before we prove the completeness theorem, we need a few more definitions.

**Definition 5.4.** A set of formulas $S$ in the variables $\{p_1, p_2, \ldots\}$ is complete if for every formula $\phi$ in the variables $\{p_1, p_2, \ldots\}$, either $\phi \in S$ or $\neg \phi \in S$.

**Definition 5.5.** A set of formulas $S$ is said to be inconsistent if there is a formula $\psi$ such that $S \vdash \psi$ and $S \vdash \neg \psi$. If $S$ is not inconsistent, then we say $S$ is consistent.

A key fact which will be given as homework is the following:

**Exercise 5.6.** If $S \cup \{\neg \phi\}$ is inconsistent, then $S \vdash \phi$.

**Theorem 5.7.** If $S = \{\phi_1, \phi_2, \ldots\}$ is a set of formulas, then if $S$ implies $\psi$ for some formula $\psi$, then $S \vdash \psi$.

**Proof.** We prove the contrapositive; if it is not the case that $S \vdash \phi$, then $S$ does not imply $\phi$.

Now to show $S$ does not imply $\phi$, we must construct a valuation that makes every formula of $S$ true, and $\phi$ false. We do this as follows.

Let $S_0 = S \cup \{\neg \phi\}$. Now we claim that this set of formulas is consistent. This is because if it were inconsistent, then by Exercise 5.6, $S \vdash \phi$ contradicting our assumption.

Now we make a sequence $S_0, S_1, \ldots$ of consistent sets of formulas such that their union $\overline{S} = \bigcup_i S_i$ is consistent and complete. Let $\theta_1, \theta_2, \ldots$ be a list of all formulas using variables occurring the the formulas of $S$.

Given $S_i$, consider $S \cup \{\neg \theta_{i+1}\}$. If this set of formulas is consistent, let $S_{i+1} = S \cup \{\neg \theta_{i+1}\}$. Otherwise, since $S_i \cup \{\neg \theta_{i+1}\}$ is inconsistent, $S_i \vdash \theta_{i+1}$, and hence $S_i \cup \{\theta_{i+1}\}$ is consistent, since any proof $S_i \cup \{\theta_{i+1}\} \vdash \xi$ can be converted into a proof $S_i \vdash \xi$. Hence, we can let $S_{i+1} = S_i \cup \{\theta_{i+1}\}$.
Now by construction, \( \mathcal{S} = \bigcup_i S_i \) is complete. We also have that \( \mathcal{S} \) is consistent, since proofs are finite, and hence any proof from \( \mathcal{S} \) must also be a proof from \( S_n \) for \( n \) large enough.

It will be a homework exercise to finish this proof by constructing a valuation satisfying \( \mathcal{S} \) and hence making every formula of \( S \) true and \( \psi \) false.