6c Lecture 7: April 21, 2015

5 Soundness and completeness of our Hilbert-style proof system

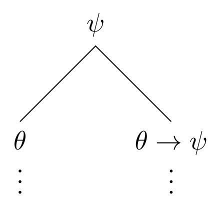
Our goal in this section is to prove the soundness and completeness of our Hilbert-style proof system. The easy half of this pair of theorems is soundness.

Theorem 5.1. *Suppose $S = \{\phi_1, \phi_2, \dots\}$ is a set of formulas, and $S \vdash \psi$. Then S implies ψ.*

Proof. By induction on the height of proofs. The guts of this proof is just just checking that each type of step we use in our proofs is logically valid.

Base case: Proofs of height 0. Such a proof is just a statement of a logical axiom or a formula in S. If ψ is a logical axiom, or $\psi = \phi_i$ for some formula ϕ_i in S, then clearly S implies ψ.

Inductive case: Suppose that proofs of height n are sound, and we have a proof that $S \vdash \psi$ of height $n+1$, whose last step must be an instance of modus ponens concluding ψ from the formulas θ and $\theta \to \psi$.



Now since there are proofs of θ and $\theta \to \psi$ using proofs of height $\leq n$, by our induction hypothesis, S implies θ and S implies $\theta \to \psi$. But then any valuation making S true makes θ true and $\theta \to \psi$ true, and hence also makes ψ true. Thus, S implies ψ. $\square$

Theorem 5.2 (The Deduction Theorem)**.** *Suppose that S is a set of formulas and A, B are formulas. If $S \cup \{A\} \vdash B$, then $S \vdash A \to B$.*

Remark 5.3. *It follows from the definition of our proof system that if $S \vdash A \to B$, then $S \cup \{A\} \vdash B$. So the Deduction Theorem tells us that these two things are equivalent.*

Proof. This is an induction on the height of the proof of B. Suppose that $S \cup \{A\} \vdash B$ with a proof of height 1. There are three things that could be happening.

1. B is a logical axiom. In this case $S \vdash B$. Also, since $B \to (A \to B)$ is an example of logical axiom 1, $S \vdash B \to (A \to B)$. Applying modus ponens, we find that $S \vdash A \to B$.

2. $B \in S$. Then $S \vdash B$ and the same argument as above works.

3. $B = A$. Then since $S \vdash A \to A$ by our example from last time, we find that $S \vdash A \to B$.

Now suppose inductively that B follows from two statements by modus ponens, i.e. there is some formula ψ such that $S \cup \{A\} \vdash \psi$ and $S \cup \{A\} \vdash \psi \to B$. By our inductive assumption, $S \vdash A \to \psi$ and $S \vdash A \to (\psi \to B)$. Using logical axiom 2, we find that

$$(A \to (\psi \to B)) \to ((A \to \psi) \to (A \to B))$$

is a logical axiom. Applying modus ponens twice, we find that $S \vdash A \to B$, as desired. □

Before we prove the completeness theorem, we need a few more definitions.

Definition 5.4. A set of formulas S in the variables $\{p_1, p_2, \ldots\}$ is *complete* if for every formula ϕ in the variables $\{p_1, p_2, \ldots\}$, either $\phi \in S$ or $\neg\phi \in S$.

Definition 5.5. A set of formulas S is said to be *inconsistent* if there is a formula ψ such that $S \vdash \psi$ and $S \vdash \neg\psi$. If S is not inconsistent, then we say S is *consistent*.

A key fact which will be given as homework is the following:

Exercise 5.6. If $S \cup \{\neg\phi\}$ is inconsistent, then $S \vdash \phi$.

Theorem 5.7. *If $S = \{\phi_1, \phi_2, \ldots\}$ is a set of formulas, then if S implies ψ for some formula ψ, then $S \vdash \psi$.*

Proof. We prove the contrapositive; if it is not the case that $S \vdash \phi$, then S does not imply ϕ.

Now to show S does not imply ϕ, we must construct a valuation that makes every formula of S true, and ϕ false. We do this as follows.

Let $S_0 = S \cup \{\neg\phi\}$. Now we claim that this set of formulas is consistent. This is because if it were inconsistent, then by Exercise 5.6, $S \vdash \phi$ contradicting our assumption.

Now we make a sequence $S_0, S_1, \ldots$ of consistent sets of formulas such that their union $\overline{S} = \bigcup_i S_i$ is consistent and complete. Let $\theta_1, \theta_2, \ldots$ be a list of all formulas using variables occurring the the formulas of S.

Given S_i, consider $S \cup \{\neg\theta_{i+1}\}$. If this set of formulas is consistent, let $S_{i+1} = S \cup \{\neg\theta_{i+1}\}$. Otherwise, since $S_i \cup \{\neg\theta_{i+1}\}$ is inconsistent, $S_i \vdash \theta_{i+1}$, and hence $S_i \cup \{\theta_{i+1}\}$ is consistent, since any proof $S_i \cup \{\theta_{i+1}\} \vdash \xi$ can be converted into a proof $S_i \vdash \xi$. Hence, we can let $S_{i+1} = S_i \cup \{\theta_{i+1}\}$.

Now by construction, $\overline{S} = \bigcup_i S_i$ is complete. We also have that $\overline{S}$ is consistent, since proofs are finite, and hence any proof from $\overline{S}$ must also be a proof from S_n for n large enough.

It will be a homework exercise to finish this proof by constructing a valuation satisfying $\overline{S}$ and hence making every formula of S true and ψ false. □