6c Lecture 6: April 16, 2014

3.1 The method of resolution

Now we will describe a method for proving that a formula in CNF is unsatisfiable. Recall that a formula in CNF has the form $(\ell_{1,1} \vee \ldots, \ell_{1,n_1}) \wedge (\ell_{1,2} \vee \ldots, \ell_{1,n_2}) \wedge \ldots \wedge (\ell_{1,k} \vee \ldots, \ell_{1,n_k})$, where the $\ell_{i,j}$ are literals. Since the order of the literals $\ell_{i,1} \ldots \ell_{i,n_i}$ does not matter inside each conjunct, we can think of them as simply being a set of literals, which we call a clause. It is this observation which motivates the following definition, which essentially is just new terminology for parts of a formula in CNF.

Definition 3.1. A *clause* is a set of literals. A clause is *satisfied* by a valuation v if at least one of the literals in the clause is true according to the valuation. A set of clauses is *satisfiable* iff there is some valuation making each clause in the set satisfied.

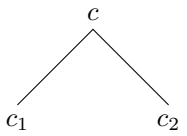
We let $\square$ denote the empty clause.

For example, the CNF formula $\{(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)\}$ is associated to the set of clauses $\{\{p, \neg q, r\}, \{\neg p, q, r\}, \{\neg p, \neg q, \neg r\}\}$, and this set of clauses is satisfiable iff the original formula is.

Now suppose we have two disjunctions of literals where some propositional variable p appears in the first, and $\neg p$ appears in the second. For example, $p \vee q \vee r$ and $\neg p \vee q \vee \neg t$. If both these disjunctions are true, then we can conclude that at least one of the remaining literals other than p and $\neg p$ is true. This is because if p is false, then q or r is true (to make the first disjunction true), and if p is true, then q or $\neg t$ is true. Hence, from $p \vee q \vee r$ and $\neg p \vee q \vee \neg t$ we can conclude $q \vee r \vee \neg t$. This idea is what leads us to make the definition of a resolvent of two clauses.

Definition 3.2. Suppose c_1 and c_2 are clauses. We say that a third clause c is a *resolvent* of c_1 and c_2 if there is a propositional variable p such that c_1 contains p, c_2 contains $\neg p$, and $c = (c_1 \setminus \{p\}) \cup (c_2 \setminus \{\neg p\})$.

We denote this by the diagram



Hence, for example:

- $\{r, \neg s, t\}$ is a resolvent of $\{\neg p, r, t\}$ and $\{p, \neg s, r, t\}$.

- The empty clause $\{\}$ is a resolvent of $\{q\}$ and $\{\neg q\}$.
- $\{p, \neg p\}$ is a resolvent of $\{p, \neg q\}$ and $\{\neg p, q\}$.
- $\{\neg q, q\}$ is a resolvent of $\{p, \neg q\}$ and $\{\neg p, q\}$.

Note that the empty clause is unsatisfiable; there is no literal in this clause which can be made true by any valuation.

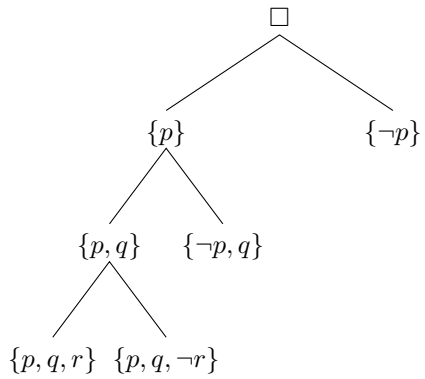
By our reasoning above, we have the following fact:

Lemma 3.3. *Suppose $\{c_1, c_2\}$ is a set of clauses and c is a resolvent of c_1 and c_2. If $\{c_1, c_2\}$ is satisfied by the valuation v, then so is c.*

Definition 3.4. A *proof by resolution* of a clause c from a set of clauses C is a finite tree whose vertices are each assigned a clause such that

1. The root is assigned c
2. If a vertex assigned c has children it has exactly two children and c is a resolvent of the clauses assigned to them.
3. If a vertex has no children it is assigned a clause from C.

We give an example of a proof by resolution of $\square$ from the set of clauses $\{\{p, q, \neg r\}, \{\neg p\}, \{p, q, r\}, \{p, \neg q\}\}$:



The following theorem is essentially soundness for proofs by resolution.

Theorem 3.5 (Soundness for proofs by resolution)**.** *If there is a proof by resolution of a clause c from a set of clauses C, then if C is satisfied by the valuation v, then so is c.*

Proof. In class, proved by induction on height of the proof. The key fact one needs to use for the inductive step is Lemma 3.3 above. □

In class we also proved the completeness theorem for proofs by resolution. We began with the following definition:

Definition 3.6. Suppose C is a set of clauses. Then we define

$$C(p = \text{True}) = \{c \setminus \{\neg p\} : c \in C \wedge p \notin c\}.$$

and

$$C(p = \text{True}) = \{c \setminus \{p\} : c \in C \wedge \neg p \notin c\}.$$

For example, if $C = \{\{p, s\}, \{\neg p, q\}, \{\neg q, s\}, \{\neg p, q, \neg s\}, \}$, then $C(p = \text{True}) = \{\{q\}, \{\neg q, s\}, \{q, \neg s\}\}$, and $C(p = \text{False}) = \{\{s\}, \{\neg q, s\}\}$.

Lemma 3.7. *Suppose C is a set of clauses. Then $C(p = True)$ is satisfiable iff there is a valuation satisfying C such that p is true. Further, $C(p = False)$ is satisfiable iff there is a valuation satisfying C such that p is false.*

Theorem 3.8 (Completeness for proofs by resolution)**.** *If C is an unsatisfiable set of clauses, then there is a proof by resolution of the empty clause from C.*

Proof. Given in class. The rough idea is to prove the theorem by induction on the number of propositional variables $\{p_1, \ldots, p_n\}$ used in the set of clauses. Assuming C is an unsatisfiable set of clauses in the variables $\{p_1, \ldots, p_n, p_{n+1}\}$, both $C(p_{n+1} = \text{True})$ and $C(p_{n+1} = \text{False})$ are unsatisfiable by Lemma 3.7.

By assumption, you then have proofs by resolution of $\square$ from $C(p_{n+1} = \text{True})$ and $C(p_{n+1} = \text{False})$. These don't work as proofs from C, but the only difference is that clauses from $C(p_{n+1} = \text{True})$ may be missing $\neg p_{n+1}$ and clauses from $C(p_{n+1} = \text{False})$ may be missing p_{n+1}. So if we put those variables back in any clauses that we removed them from, the worst-case scenario is that we now have proofs from C of p_{n+1} and $\neg p_{n+1}$. But then one more resolution gets us $\square$. $\square$

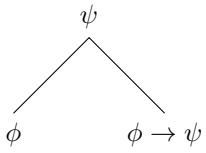
3.2 A Hilbert-style formal proof system

One disadvantage of proofs of resolution is that they bear little resemblance to the mathematical proofs we generally use in practice. Next, we describe a formal proof system which is closer to the proof we usually use in mathematics. The Hilbert-style system we will describe has 3 logical axioms and 1 rule. This system deals only with proofs of formulas involving the connectives $\neg$ and $\rightarrow$.

Definition 3.9. For any formulas ϕ, ψ, θ, the following are logical axioms:

1. $\phi \rightarrow (\psi \rightarrow \phi)$.
2. $(\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta))$
3. $(\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$.

In addition, we have the rule (modus ponens) that if ϕ and $\phi \rightarrow \psi$ are true, then ψ is true, which we represent with the diagram:



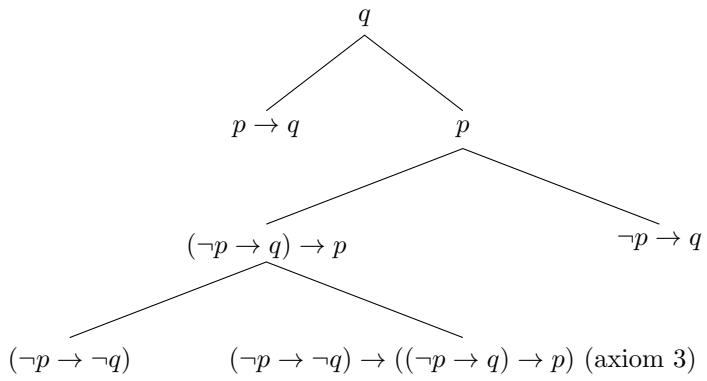
Now we can define a formal proof in this system as follows:

Definition 3.10. A *Hilbert-style proof* of a formula θ from a set of formulas S is a finite tree whose vertices are each assigned a formula such that

1. The root is assigned θ
2. If a vertex assigned ψ has children it has exactly two children, one is assigned some formula ϕ and the other is assigned $\phi \to \psi$ (so this corresponds to an instance of modus ponens).
3. If a vertex has no children it is assigned a formula from S or a logical axiom.

If there is such a proof θ from S, we write $S \vdash \theta$.

We give an example showing that $\{\neg p \to q, \neg p \to \neg q, p \to q\} \vdash q$



We give another example showing the from the empty set of formulas we can prove $p \to p$ (and similarly we can prove $\phi \to \phi$ for every formula ϕ).