HOMEWORK 6.

Homework is due on Wednesday the 20th of May at 12:00pm.

While collaboration is encouraged, you must write your own solutions.

13.2.18 Let $K$ be a field and let $K(x)$ be the field of rational functions in $x$ with coefficients from $K$. Let $t \in K(x)$ be the rational function $P(x)/Q(x)$ with relatively prime polynomials $P(x), Q(x) \in K[x]$, with $Q(x) \neq 0$. Then $K(x)$ is an extension of $K(t)$ and to compute its degree it is necessary to compute the minimal polynomial with coefficients in $K(t)$ satisfied by $x$.

(a) Show that the polynomial $P(X) - tQ(X)$ in the variable $X$ is and coefficients in $K(t)$ is irreducible over $K(t)$ and has $x$ as its root. [See hint]

(b) Show that the degree of $P(X) - tQ(X)$ as a polynomial in $X$ with coefficients in $K(t)$ is the maximum of the degrees of $P(x)$ and $Q(x)$.

Note: There is no need to do part (c) of 13.2.18 in the book as it should be obvious that it follows from (a) and (b).

14.7.3 Let $F$ be a field of characteristic $\neq 2$. State and prove a necessary and sufficient condition on $\alpha, \beta \in F$ so that $F(\sqrt{\alpha}) = F(\sqrt{\beta})$. Use this to determine whether $Q(\sqrt{1 - \sqrt{2}}) = Q(i, \sqrt{2})$.

Question 3 Give an example (with justification) of a polynomial whose Galois group over $\mathbb{Q}$ is the symmetric group on 6 elements, i.e., $S_6$.

14.9.10 Prove that a purely transcendental proper extension of a field is never algebraically closed.

Question 5 Show that if $t$ is transcendental over $\mathbb{Q}$ then $\mathbb{Q}(t, \sqrt{1 - t^2})$ is a purely transcendental extension of $\mathbb{Q}$.