Chapter 6

Green’s Theorem in the Plane

Recall the following special case of a general fact proved in the previous chapter. Let $C$ be a piecewise $C^1$ plane curve, i.e., a curve in $\mathbb{R}^2$ defined by a piecewise $C^1$-function

$$\alpha : [a, b] \to \mathbb{R}^2$$

with end points $\alpha(a)$ and $\alpha(b)$. Then for any $C^1$ scalar field $\varphi$ defined on a connected open set $U$ in $\mathbb{R}^2$ containing $C$, we have

$$\int_C \nabla \varphi \cdot d\alpha = \varphi(\alpha(b)) - \varphi(\alpha(a)).$$

In other words, the integral of the gradient of $\varphi$ along $C$, in the direction stipulated by $\alpha$, depends only on the difference, measured in the right order, of the values of $\varphi$ on the end points. Note that the two-point set $\{a, b\}$ is the boundary of the interval $[a, b]$ in $\mathbb{R}$. A general question which arises is to know if similar things hold for integrals over surfaces and higher dimensional objects, i.e., if the integral of the analog of a gradient, sometimes called an exact differential, over a geometric shape $M$ depends only on the integral of its primitive on the boundary $\partial M$.

Our object in this chapter is to establish the simplest instance of such a phenomenon for plane regions. First we need some preliminaries.

6.1 Jordan curves

Recall that a curve $C$ parametrized by a continuous $\alpha : [a, b] \to \mathbb{R}^2$ is said to be closed iff $\alpha(a) = \alpha(b)$. It is called a Jordan curve, or a simple
A closed curve, iff $\alpha$ is piecewise $C^1$ and 1-1 (injective) except at the end points (i.e. $x \neq y$ implies $\alpha(x) \neq \alpha(y)$ or $\{x,y\} = \{a,b\}$). Geometrically, this means the curve doesn’t cross itself. Examples of Jordan curves are circles, ellipses, and in fact all kinds of ovals. The hyperbola defined by $\alpha : [0, 1] \to \mathbb{R}^2$, $x \mapsto \frac{c}{x}$, is (for any $c \neq 0$) simple, i.e., it does not intersect itself, but it is not closed. On the other hand, the clover is closed, but not simple.

Here is a fantastic result, in some sense more elegant than the ones in Calculus we are trying to establish, due to the French mathematician Camille Jordan; whence the name Jordan curve.

**Theorem.** Let $C$ be a Jordan curve in $\mathbb{R}^2$. Then there exists connected open sets $U, V$ in the plane such that

(i) $U, V, C$ are pairwise mutually disjoint,

and

(ii) $\mathbb{R}^2 = U \cup V \cup C$.

In other words, any Jordan curve $C$ separates the plane into two disjoint, connected regions with $C$ as the common boundary. Such an assertion is obvious for an oval but not (at all) in general. Note that the theorem applies to any continuous simple closed curve, it does not need the curve to be piecewise $C^1$. That is also why the proof properly belongs to an area of mathematics called "topology", not "calculus", the subject of this class. In any case the proof involves some effort and is too long to give in this course. For an interesting account of the history as well as of Jordan’s original proof see "Thomas Hales, Jordan’s Proof of the Jordan Curve Theorem, Studies in Logic, Grammar and Rhetoric 10 (23) 2007, http://mizar.org/trybulec65/4.pdf".

The two regions $U$ and $V$ are called the interior and exterior of $C$. To distinguish which is which let’s prove

**Lemma:** In the above situation exactly one of $U$ and $V$ is bounded. This is called the interior of $C$.

**Proof.** Since $[a,b]$ is compact and $\alpha$ is continuous the curve $C = \alpha([a,b])$ is compact, hence closed and bounded. Pick a disk $D(r)$ of some large radius $r$ containing $C$. Then $S := \mathbb{R}^2 \setminus D(r) \subseteq U \cup V$. Clearly $S$ is connected, so any two points $P, Q \in S$ can be joined by a continuous path $\beta : [0, 1] \to S$ with $P = \beta(0)$, $Q = \beta(1)$. We have $[0, 1] = \beta^{-1}(U) \cup \beta^{-1}(V)$ since $S$ is covered by $U$ and $V$. Since $\beta$ is continuous the sets $\beta^{-1}(U)$ and $\beta^{-1}(V)$ are open.
subsets of \([0, 1]\). If \(P \in U\), say, put \(t_0 = \sup\{t \in [0, 1] | \beta(t) \in U\} \in [0, 1]\). If \(\beta(t_0) \in U\) then, since \(\beta^{-1}(U)\) is open, we find points in a small neighborhood of \(t_0\) mapped to \(U\). If \(t_0 < 1\) this would mean we’d find points bigger than \(t_0\) mapped into \(U\) which contradicts the definition of \(t_0\). So if \(\beta(t_0) \in U\) then \(t_0 = 1\) and \(Q = \beta(1) \in U\). If, on the other hand, \(\beta(t_0) \in V\), there is an interval of points around \(t_0\) mapped to \(V\) which also contradicts the definition of \(t_0\) (we’d find a smaller upper bound in this case). So the only conclusion is that if one point \(P \in S\) lies in \(U\) so do all other points \(Q\). Then \(S \subseteq U\) and \(V \subseteq D(r)\) so \(V\) is bounded.

Recall that a parametrized curve has an orientation. A Jordan curve can either be oriented counterclockwise or clockwise. We usually orient a Jordan curve \(C\) so that the interior, \(V\) say, lies to the left as we traverse the curve, i.e. we take the counterclockwise orientation. This is also called the positive orientation. In fact we could define the counterclockwise or positive orientation by saying that the interior lies to the left.

Note finally that the term “interior” is a bit confusing here as \(V\) is not the set of interior points of \(C\); but it is the set of interior points of the union of \(V\) and \(C\).

### 6.2 Simply connected regions

A connected open set \(R\) in the plane is said to be simply connected if every Jordan curve \(C\) in \(R\) can be continuously deformed to a point in \(R\). Here is the rigorous definition using the Jordan curve theorem.

**Definition.** A connected open set \(R \subseteq \mathbb{R}^2\) is called simply connected (s.c.) if for any Jordan curve \(C \subset R\), the interior of \(C\) lies completely in \(R\).

**Examples of s.c. regions:**

1. \(R = \mathbb{R}^2\)
2. \(R = \text{interior of a Jordan curve } C\).

The simplest case of a non-simply connected plane region is the annulus given by \(\{v \in \mathbb{R}^2 | c_1 < \|v\| < c_2\}\) with \(0 < c_1 < c_2\).

When a region is not simply connected, one often calls it multiply connected or m.c.
6.3 Green’s theorem for simply connected plane regions

Recall that if \( f \) is a vector field with image in \( \mathbb{R}^n \), we can analyze \( f \) by its coordinate fields \( f_1, f_2, \ldots, f_n \), which are scalar fields. When \( n = 2 \) (resp. \( n = 3 \)), it is customary notation to use \( P, Q \) (resp. \( P, Q, R \)) instead of \( f_1, f_2 \) (resp. \( f_1, f_2, f_3 \)).

**Theorem 1.** (Green) Let \( f = (P, Q) \) be a \( C^1 \) vector field on an open set \( \mathcal{D} \) in \( \mathbb{R}^2 \). Suppose \( C \) is a piecewise \( C^1 \) Jordan curve with interior \( U \) such that \( \Phi := C \cup U \) lies entirely in \( \mathcal{D} \). Then we have

\[
\iint_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C} f \cdot d\alpha.
\]

Here \( C \) is the boundary \( \partial \Phi \) of \( \Phi \), and moreover, the integral over \( C \) is taken in the positive direction. Note that

\[
f \cdot d\alpha = (P(x, y), Q(x, y)) \cdot (x'(t), y'(t)) \, dt,
\]

and so one sometimes also uses the following notation

\[
\oint_{C} f \cdot d\alpha = \oint_{C} (P \, dx + Q \, dy).
\]

This beautiful theorem combined with Theorem 3 from the last chapter has the following immediate

**Corollary 1.** Let \( f = (P, Q) \) be a \( C^1 \) vector field on a simply connected open set \( \mathcal{D} \) in \( \mathbb{R}^2 \). Then

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
\]

if and only if

\[
\oint_{C} f \cdot d\alpha = 0
\]

for all Jordan curves \( C \subseteq \mathcal{D} \).
Remark. One can show that for simply connected $D \subseteq \mathbb{R}^2$ the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ is in fact equivalent to the vector field $f = (P, Q)$ being conservative, i.e. the vanishing of $\oint_C f \cdot d\alpha$ for all closed curves $C$. One possible proof of this fact uses step-paths and can be found in the textbook.

Given any $C^1$ Jordan curve $C$ with $\Phi = C \cup U$ as above, we can try to finely subdivide the region using $C^1$ arcs such that $\Phi$ is the union of sub-regions $\Phi_1, \ldots, \Phi_r$ with Jordan curves as boundaries and with non-intersecting interiors, such that each $\Phi_j$ is simultaneously a region of type $I$ and $II$ (see chapter 5). This can almost always be achieved. So we will content ourselves, mostly due to lack of time and preparation on Jordan curves, with proving the theorem only for regions which are of type $I$ and $II$.

Theorem A follows from the following, seemingly more precise, identities:

(i) $\iint_{\Phi} \frac{\partial P}{\partial y} dxdy = -\oint_C Pdx$

(ii) $\iint_{\Phi} \frac{\partial Q}{\partial x} dxdy = \oint_C Qdy$.

Now suppose $\Phi$ is of type $I$, i.e., of the form $\{a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with $a < b$ and $\varphi_1, \varphi_2$ continuously differentiable on $[a, b]$. Then the boundary $C$ has 4 components given by

$C_1 : \text{graph of } \varphi_1(x), a \leq x \leq b$

$C_2 : \text{ } x = b, \varphi_1(b) \leq y \leq \varphi_2(b)$

$C_3 : \text{graph of } \varphi_2(x), a \leq x \leq b$

$C_4 : \text{ } x = a, \varphi_1(a) \leq y \leq \varphi_2(a)$.

As $C$ is positively oriented, each $C_i$ is oriented as follows: In $C_1$, traverse from $x = a$ to $x = b$; in $C_2$, traverse from $y = \varphi_1(b)$ to $y = \varphi_2(b)$; in $C_3$, go from $x = b$ to $x = a$; an in $C_4$, go from $\varphi_2(a)$ to $\varphi_1(a)$.

It is easy to see that

$$\int_{C_2} Pdx = \int_{C_4} Pdx = 0,$$

since $C_2$ and $C_4$ are vertical segments allowing no variation in $x$. Hence we have

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\begin{equation}
- \int_C P \, dx = \left( \int_{C_1} P \, dx + \int_{C_3} P \, dx \right) = \int_a^b \left[ P(x, \phi_2(x)) - P(x, \phi_1(x)) \right] \, dx.
\end{equation}

On the other hand, since \( f \) is a \( C^1 \) vector field, \( \frac{\partial P}{\partial y} \) is continuous, and since \( \Phi \) is a region of type \( I \), we may apply Fubini’s theorem and get

\begin{equation}
\int \int_{\Phi} \frac{\partial P}{\partial y} \, dx \, dy = \int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}.
\end{equation}

Note that \( x \) is fixed in the inside integral on the right. We see, by the fundamental theorem of 1-variable Calculus, that

\begin{equation}
\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y} = P(x, \phi_2(x)) - P(x, \phi_1(x)).
\end{equation}

Putting together (1), (2) and (3), we immediately obtain identity (i).

Similarly (ii) can be seen to hold by exploiting the fact that \( \Phi \) is also of type \( II \).

Hence Theorem A is now proved for a region \( \Phi \) which is of type \( I \) and \( II \).

### 6.4 An area formula

The example below will illustrate a very useful consequence of Green’s theorem, namely that the area of the inside of a \( C^1 \) Jordan curve \( C \) can be computed as

\[ A = \frac{1}{2} \oint_C (x \, dy - y \, dx). \quad (*) \]

A proof of this fact is easily supplied by taking the vector field \( f = (P, Q) \) in Theorem A to be given by \( P(x, y) = -y \) and \( Q(x, y) = x \). Clearly, \( f \) is \( C^1 \) everywhere on \( \mathbb{R}^2 \), and so the theorem is applicable. The identity for \( A \) follows as \( \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} = 1 \).
Example:

Fix any positive number $r$, and consider the hypocycloid $C$ parametrized by

$$
\alpha : [0, 2\pi] \rightarrow \mathbb{R}^2, \ t \mapsto (r \cos^3 t, r \sin^3 t).
$$

Then $C$ is easily seen to be a piecewise $C^1$ Jordan curve. Note that it is also given by $x^\frac{2}{3} + y^\frac{2}{3} = r^\frac{2}{3}$. We have

$$
A = \frac{1}{2} \int_C (xdy - ydx) = \frac{1}{2} \int_0^{2\pi} (xy'(t) - yx'(t)) dt.
$$

Now, $x'(t) = 3r \cos^2 t (\sin t)$, and $y'(t) = 3r \sin^2 t \cos t$. Hence

$$
xy'(t) - yx'(t) = (r \cos^3 t)(3r \sin^2 t \cos t) + (r \sin^3 t)(3r \cos^2 t \sin t)
$$

which simplifies to $3r^2 \sin^2 t \cos^2 t$, as $\sin^2 t + \cos^2 t = 1$. So we obtain

$$
A = \frac{3r^2}{2} \int_0^{2\pi} \left( \frac{\sin 2t}{2} \right)^2 dt = \frac{3r^2}{8} \int_0^{2\pi} \left( \frac{1 - \cos 4t}{2} \right)^2 dt,
$$

by using the trigonometric identities $\sin 2u = 2 \sin u \cos u$ and $\cos 2u = 1 - 2 \sin^2 u$. Finally, we then get

$$
A = \frac{2r^2}{16} \left[ \int_0^{2\pi} (1 - \cos 4t) dt \right] = \frac{3r^2}{16} \left[ \frac{t - \sin 4t}{4} \right]_0^{2\pi}
$$

i.e.,

$$
A = \frac{3\pi r^2}{8}.
$$

Remark. Using (*) as well as some Fourier analysis (Wirtinger inequality) one can derive the isoperimetric inequality

$$
4\pi A \leq L^2
$$

where $L = \int_C ds$ is the length of the Jordan curve $C$. 

6.5 Green’s theorem for multiply connected regions

We mentioned earlier that the annulus in the plane is the simplest example of a non-simply connected region. But it is not hard to see that we can cut this region into two pieces, each of which is the interior of a $C^1$ Jordan curve. We may then apply Theorem A of §3 to each piece and deduce statement over the annulus as a consequence. To be precise, pick any point $z$ in $\mathbb{R}^2$ and consider

$$\Phi = \bar{B}_z(r_2) - \bar{B}_z(r_1),$$

for some real numbers $r_1, r_2$ such that $0 < r_1 < r_2$. Here $\bar{B}_z(r_i)$ denotes the closed disk of radius $r_i$ and center $z$.

Let $C_1$, resp. $C_2$, denote the positively oriented (circular) boundary of $\bar{B}_z(r_1)$, resp. $\bar{B}_z(r_2)$. Let $D_i$ be the flat diameter of $\bar{B}_z(r_i)$, i.e., the set $\{x_0 - r_i \leq x \leq x_0 + r_i, \ y = y_0\}$, where $x_0$ (resp. $y_0$) denotes the $x$-coordinate (resp. $y$-coordinate) of $z$. Then $D_2 \cap \Phi$ splits as a disjoint union of two horizontal segments $D_+ = D_2 \cap \{x > x_0\}$ and $D_- = D_2 \cap \{x < x_0\}$. Denote by $C_i^+$ (resp. $C_i^-$) the upper (resp. lower) half of the circle $C_i$, for $i = 1, 2$. Then $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+$ (resp. $\Phi^-$) is the region bounded by $C^+ = C_2^+ \cup D_- \cup C_1^+ \cup D_+$ (resp. $C^- = C_2^- \cup D_+ \cup D_1^- \cup D_-\$). We can orient the piecewise $C^1$ Jordan curves $C^+$ and $C^-$ in the positive direction. Let $U^+, U^-$ denote the interiors of $C^+, C^-$. Then $U^+ \cap U^- = \emptyset$, and $\Phi^\pm = C^\pm \cup U^\pm$.

Now let $f = (P, Q)$ be a $C^1$-vector field on a connected open set containing $\bar{B}_z(r_2)$. Then, combining what we get by applying Green’s theorem for s.c. regions to $\Phi^+$ and $\Phi^-$, we get:

$$\iint_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C_2} (P \, dx + Q \, dy) - \oint_{C_1} (P \, dx + Q \, dy), \quad (*)$$

where both the line integrals on the right are taken in the positive (counterclockwise) direction. Note that the minus sign in front of the second line integral on the right comes from the need to orient $C^\pm$ positively.

In some sense, this is the key case to understand as any multiply connected region can be broken up into a finite union of shapes each of which can be continuously deformed to an annulus. Arguing as above, we get the following (slightly stronger) result:
Theorem 2. (Green's theorem for m.c. plane regions) Let $C_1, C_2, \ldots, C_r$ be non-intersecting piecewise $C^1$ Jordan curves in the plane with interiors $U_1, U_2, \ldots, U_r$ such that

(i) $U_1 \supset C_i, \forall i \geq 2$,

(ii) $U_i \cap U_j = \emptyset$, for all $i, j \geq 2$.

Put $\Phi = C_1 \cup U_1 - \bigcup_{i=2}^{r} U_i$, which is not simply connected if $r \geq 2$. Also let $f = (P, Q)$ be a $C^1$ vector field on a connected open set $S$ containing $\Phi$. Then we have

$$
\iint_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C_1} (P \, dx + Q \, dy) - \sum_{i=2}^{r} \oint_{C_i} (P \, dx + Q \, dy)
$$

where each $C_j, j \geq 1$ is positively oriented.

Corollary 2. Let $C_1, \ldots, C_r, f$ be as in Theorem B. In addition, suppose that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere on $S$. Then we have

$$
\oint_{C_1} (P \, dx + Q \, dy) = \sum_{i=2}^{r} \oint_{C_i} (P \, dx + Q \, dy).
$$

6.6 The winding number

Let $C$ be an oriented piecewise $C^1$ curve in the plane, and let $z_0 = (x_0, y_0)$ be a point not lying on $C$. Then the winding number of $C$ relative to $z_0$ is defined to be

$$
W(C, z_0) := \frac{1}{2\pi} \oint_{C} \left( -\frac{y - y_0}{r^2} \, dx + \frac{x - x_0}{r^2} \, dy \right),
$$

where $r = \| (x, y) - z_0 \| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

When $C$ is parametrized by a piecewise $C^1$ function $\alpha : [a, b] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$, then it is easy to see that

$$
W(C, z_0) = \frac{1}{2\pi} \int_{a}^{b} \frac{(x(t) - x_0)y'(t) - (y(t) - y_0)x'(t)}{(x(t) - x_0)^2 + (y(t) - y_0)^2} \, dt
$$
Some write $W(\alpha, z_0)$ instead of $W(C, z_0)$.

However, the geometric meaning of this particular integral becomes much clearer if one uses polar coordinates around the point $(x_0, y_0)$. This means that $\alpha(t)$ is now a function of the form

$$(x(t), y(t)) = (x_0 + r(t) \cos(\phi(t)), y_0 + r(t) \sin(\phi(t)))$$

for certain piecewise $C^1$-functions $r(t)$ and $\phi(t)$. Then

$$(x(t) - x_0)y'(t) = r(t) \cos(\phi(t))\left(r'(t) \sin(\phi(t)) + r(t) \cos(\phi(t))\phi'(t)\right)$$

$$(y(t) - y_0)x'(t) = r(t) \sin(\phi(t))\left(r'(t) \cos(\phi(t)) - r(t) \sin(\phi(t))\phi'(t)\right)$$

and

$$(x(t) - x_0)y'(t) - (y(t) - y_0)x'(t) = r(t)^2\phi'(t)$$

so that

$$W(C, z_0) = \frac{1}{2\pi} \int_a^b \phi'(t) dt = \frac{1}{2\pi} (\phi(b) - \phi(a)).$$

It is clear that this number is an integer if $C$ is a closed curve. Indeed, if $\alpha(a) = \alpha(b)$ then $(\cos(\phi(a)), \sin(\phi(a))) = (\cos(\phi(b)), \sin(\phi(b)))$ which holds if and only if $\phi(a) - \phi(b) = 2\pi k$ with $k \in \mathbb{Z}$.

This proves the first half of the following useful result:

**Proposition 1.** Let $C$ be a piecewise $C^1$ closed curve in $\mathbb{R}^2$, and let $z_0 \in \mathbb{R}^2$ be a point not meeting $C$.

(a) $W(C, z_0) \in \mathbb{Z}$.

(b) If $C$ is a Jordan curve then $W(C, z_0) \in \{0, 1, -1\}$.

More precisely, in the case of a Jordan curve, $W(C, z_0)$ is 0 if $z_0$ is outside $C$, and it equals $\pm 1$ if $z_0$ is inside $C$.

**Proof.** Let $C$ be a Jordan curve and $z_0$ a point in the interior of $C$. One checks that the vector field $f = (P, Q) = (-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2})$ satisfies $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere on $\mathbb{R}^2 - \{z\}$. So Theorem 2 implies that the integral of $f$ over $C$ equals the negative of the integral of $f$ over a small circle around $z_0$. But this last integral one directly computes to be $\pm 1$. 

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If $z_0$ is an exterior point the integral will be equal to 0 by a similar argument. Done.

If $\bar{C}$ denotes the curve obtained from $C$ by reversing the orientation, then $W(\bar{C}, z_0) = -W(C, z_0)$. So the winding number allows us to finally give a rigorous definition of "positively oriented". The Jordan curve $C$ is positively oriented if $W(C, z_0) = 1$ for every point $z_0$ inside $C$ (i.e. $z_0$ lies in the bounded open set $U$ in the Jordan curve theorem).

A fun exercise will be to exhibit, for each integer $n$, a piecewise $C^1$, closed curve $C$ and a point $z_0$ not lying on it, such that $W(C, z_0) = n$. Of course this cannot happen if $C$ is a Jordan curve if $n \neq 0, \pm 1$. 