Chapter 4

Multiple Integrals

4.1 Basic notions

We will first discuss the question of integrability of bounded functions on closed rectangular boxes, and then move on to integration over slightly more general regions.

Recall that in one variable calculus, the integral of a function over an interval \([a, b]\) was defined as the limit, when it exists, of certain sums over finite partitions \(P\) of \([a, b]\) as \(P\) becomes finer and finer. To try to transport this idea to higher dimensions, we need to generalize the notions of partition and refinement.

In this chapter, \(R\) will always denote a closed rectangular box in \(\mathbb{R}^n\), written as

\[ R = [a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n], \]

where \(a_j, b_j \in \mathbb{R}\) for all \(j\) with \(a_j < b_j\). Every such box has a volume defined by

\[ \text{vol}(R) = (b_1 - a_1) \cdots (b_n - a_n). \]

**Definition.** A partition of \(R\) is a finite collection \(P\) of subrectangular (closed) boxes \(S_1, S_2, \ldots, S_r \subseteq R\) such that

(i) \(R = \bigcup_{j=1}^r S_j\), and

(ii) the interiors of \(S_i\) and \(S_j\) have no intersection for all \(i \neq j\).

**Definition.** A refinement of a partition \(P = \{S_j\}_{j=1}^r\) of \(R\) is another partition \(P' = \{S'_k\}_{k=1}^m\) with each \(S'_k\) contained in some \(S_j\).
Since the intersection of two rectangular subboxes is again a rectangular subbox, given any two partitions $P, P'$ of $R$, we can find a third partition $P''$ which is simultaneously a refinement of $P$ and of $P'$.

Now let $f$ be a bounded function on $R$, and let $P = \{S_j\}_{j=1}^r$ be a partition of $R$. Then $f$ is certainly bounded on each $S_j$, i.e. $f(S_j)$ is a bounded subset of $\mathbb{R}$. It was proved in Chapter 1 that every bounded subset of $\mathbb{R}$ admits a sup (lowest upper bound) and an inf (greatest lower bound).

**Definition.** The upper (resp. lower) sum of $f$ over $R$ relative to the partition $P = \{S_j\}_{j=1}^r$ is given by

$$U(f, P) = \sum_{j=1}^r \text{vol}(S_j) \sup(f(S_j))$$

(resp. $L(f, P) = \sum_{j=1}^r \text{vol}(S_j) \inf(f(S_j))$).

Here $\text{vol}(S_j)$ denotes the volume of $S_j$. Of course, we have

$$L(f, P) \leq U(f, P)$$

for all $P$.

More importantly, it is clear from the definition that if $P' = \{S'_k\}_{k=1}^m$ is a refinement of $P$, then

$$L(f, P) \leq L(f, P') \quad \text{and} \quad U(f, P') \leq U(f, P).$$

Put

$$\mathcal{L}(f) = \{L(f, P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}$$

and

$$\mathcal{U}(f) = \{U(f, P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}.$$ 

**Lemma 1** $\mathcal{L}(f)$ admits a sup, denoted $I(f)$, and $\mathcal{U}(f)$ admits an inf, denoted $I(f)$.

**Proof.** Thanks to the discussion in Chapter 1, all we have to do is show that $\mathcal{L}(f)$ (resp. $\mathcal{U}(f)$) is bounded from above (resp. below). So we will be done if we show that given any two partitions $P, P'$ of $R$, we have $L(f, P) \leq U(f, P')$ as then $\mathcal{L}(f)$ will have $U(f, P')$ as an upper bound and $\mathcal{U}(f)$ will have $L(f, P)$ as a lower bound. Choose a third partition $P''$ which refines $P$ and $P'$ simultaneously. Then we have $L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P')$. Done.
**Definition.** A bounded function $f$ on $R$ is **integrable** iff $I(f) = \overline{I}(f)$. When such an equality holds, we will simply write $I(f)$ (or $I_R(f)$ if the dependence on $R$ needs to be stressed) for $I(f) (= \overline{I}(f))$, and call it the **integral of $f$ over $R$**. Usually we will write $I(f) = \int_R f$ or $\int_{\ldots}^{R} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n$.

The notation with or without the $dx_i$ is completely equivalent. Later when we discuss change of variable formulas it will be convenient to keep the $dx_i$. Clearly, when $n = 1$, we get the integral we are familiar with, often written as $\int_{a_1}^{b_1} f(x_1) \, dx_1$.

The obvious question now is to ask if there are integrable functions. One such example is given by the **constant function** $f(x) = c$, for all $x \in R$. Then for any partition $P = \{S_j\}$, we have

$$L(f, P) = U(f, P) = c \sum_{j=1}^{r} \text{vol}(S_j) = c \text{vol}(R).$$

So $I(f) = \overline{I}(f)$ and $\int_R f = c \text{vol}(R)$.

This can be jazzed up as follows.

**Definition.** A **step function on $R$** is a bounded function $f$ on $R$ which is constant on the interior of each of the subrectangular boxes $S_j$ of some partition $P$.

**Lemma 2** Every step function $f$ on $R$ is integrable.

**Proof.** By definition, there exists a partition $P = \{S_j\}_{j=1}^{r}$ of $R$ and scalars $\{c_j\}$ such that $f(x) = c_j$ if $x \in \text{Int}(S_j)$. Then for each $\epsilon > 0$ there is a refinement $P'$ of $P$, with $L(f, P'), U(f, P')$ within $\epsilon$ of $\sum_{j=1}^{r} c_j \text{vol}(S_j)$. Hence, $I(f) = \overline{I}(f) = \sum_{j=1}^{r} c_j \text{vol}(S_j)$. Done.

### 4.2 Integrability of continuous functions

The most important bounded functions on $R$ are continuous functions. Recall from Chapter 1 that every continuous function on a compact set is bounded, and that $R$ is compact. The first result of this chapter is given by the following

**Theorem 1** Every continuous function $f$ on a closed rectangular box $R$ is integrable.
Proof. Let $S$ be any closed rectangular box contained in $R$. Define the span of $f$ on $S$ to be
\[ \text{span}_f(S) = \sup(f(S)) - \inf(f(S)). \]
A basic result about the span of continuous functions is given by

**Theorem 2 (The Small Span Theorem)** For every $\varepsilon > 0$, there exists a partition $P = \{S_j\}_{j=1}^r$ of $R$ such that $\text{span}_f(S_j) < \varepsilon$, for each $j \in \{1, ..., r\}$.

Let us first see how this implies the integrability of $f$ over $R$. Recall that it suffices to show that, given any $\varepsilon > 0$, there is a partition $P$ of $R$ such that $U(f, P) - L(f, P) < \varepsilon$. Now by the small span theorem, we can find a partition $P = \{S_j\}$ such that $\text{span}_f(S_j) < \varepsilon'$, for all $j$, where $\varepsilon' = \varepsilon / \text{vol}(R)$. Then clearly,
\[ U(f, P) - L(f, P) < \varepsilon' \text{ vol}(R) = \varepsilon. \]
Done.

It now remains to supply a proof of the small span theorem. We will prove this by contradiction. Suppose the theorem is false. Then there exists $\varepsilon_0 > 0$ such that, for every partition $P = \{S_j\}$ of $R$, $\text{span}_f(S_j) \geq \varepsilon_0$ for some $j$. For simplicity of exposition, we will only treat the case of a rectangle $R = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$. The general case is very similar, and can be easily carried out along the same lines with a bit of book-keeping. Divide $R$ into four rectangles by subdividing along the bisectors of $[a_1, b_1]$ and $[a_2, b_2]$. Then for one of these four rectangles, call it $R_1$, we must have that for every partition $\{S_j\}$ of $R_1$ there is a $j$ so that $\text{span}_f(S_j) \geq \varepsilon_0$. Do this again and again, and we finally end up with an infinite sequence of nested closed rectangles
\[ R = R_0 \supset R_1 \supset R_2 \ldots \]
such that, for every $m \geq 0$, the span of $f$ is at least $\varepsilon_0$ on some $S_{j,m}$ of any given partition of $P_m = \{S_{j,m}\}$ of $R_m$. Note that the diameter
\[ \text{dia}(R_m) = \sup\{||x - y|| \mid x, y \in R_m\} \]
is bounded by a constant $c$ times $2^{-m}$. So any sequence of points $x_m \in R_m$ will be a Cauchy sequence, and if we denote by $x = \lim_{m \to \infty} x_m$ its limit, then $x$ will lie in each $R_m$ because $R_m$ is closed and the $R_m$ are nested. Since $f$ is continuous at $x$ we can find a ”$\delta$-small” closed rectangular box $S$ with center $x$ so that $f(S) \subseteq \{t \in \mathbb{R} \mid |t - f(x)| < \varepsilon_0/2\}$, i.e. $\text{span}_f(S) < \varepsilon_0$. But $R_m$ will have to lie inside $S$ if we make $m$ so large that
\[ \text{dia}(R_m) < \frac{c}{2^m} < \frac{\text{dia}(S)}{2}. \]
So now we have a cover of $R_m$ by a single box $R_m$ on which $\text{span}_f(R_m) \leq \text{span}_f(S) < \varepsilon_0$. Contradiction! Done.
4.3 Functions with discontinuities of content zero

One is very often interested in being able to integrate bounded functions over $\mathbb{R}$ which are continuous except on a very “small” subset. To be precise, we say that a subset $Y$ of $\mathbb{R}^n$ has content zero if, for every $\varepsilon > 0$, we can find closed rectangular boxes $Q_1, \ldots, Q_m$ such that

(i) $Y \subseteq \bigcup_{i=1}^{m} Q_i$, and 
(ii) $\sum_{i=1}^{m} \text{vol}(Q_i) < \varepsilon$.

Examples. (1) A finite set of points in $\mathbb{R}^n$ has content zero. (Proof is obvious!) Exercise: Find a bounded countable set that does not have content zero.

(2) Any subset $Y$ of $\mathbb{R}$ which contains a non-empty open interval $(a, b)$ does not have content zero.

Proof. It suffices to prove that $(a, b)$ has non-zero content for $a < b$ in $\mathbb{R}$. Suppose $(a, b)$ is covered by a finite union of closed intervals $I_i$, $1 \leq i \leq m$ in $\mathbb{R}$. Then clearly, $S := \sum_{i=1}^{m} \text{length}(I_i) \geq \text{length}(a, b) = b - a$. So we can never make $S$ less than $b - a$. Done.

(3) The line segment $L = \{(x, 0) \mid a_1 < x < b_1\}$ in $\mathbb{R}^2$ has content zero. (Comparing with (2), we see that the notion of content is very much dependent on what the ambient space is, and not just on the set.)

Proof. For any $\varepsilon > 0$, cover $L$ by the single closed rectangle 

$$R = \left\{(x, y) \mid a_1 \leq x \leq b_1, -\frac{\varepsilon}{4(b_1 - a_1)} \leq y \leq \frac{\varepsilon}{4(b_1 - a_1)} \right\}.$$

Then $\text{vol}(R) = (b_1 - a_1) \frac{\varepsilon}{2(b_1 - a_1)} = \frac{\varepsilon}{2} < \varepsilon$, and we are done.

The third example leads one to ask if any bounded curve in the plane has content zero. The best result we can prove here is the following

**Proposition 1** Let $\varphi : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the graph $\Gamma$ of $\varphi$ has content zero. Similarly for $\varphi : [a, b] \subseteq \mathbb{R}^n \to \mathbb{R}$.

Proof. Note that $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y = \varphi(x)\}$. Let $\varepsilon > 0$. By the small span theorem, we can find a partition $a = t_0 < t_1 < \cdots < t_r = b$ of $[a, b]$ such that $\text{span}_{\varphi}([t_{i-1}, t_i]) < \frac{\varepsilon}{(b-a)}$, for every $i = 1, \ldots, r$. Thus the piece of $\Gamma$ lying between $x = t_{i-1}$ and $x = t_i$ can be enclosed in a closed rectangle $S_i$ of area less than $\frac{\varepsilon(t_i-t_{i-1})}{(b-a)}$. 

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Now consider the collection \( \{S_i\}_{1 \leq i \leq r} \) which covers \( \Gamma \). Then we have

\[
\sum_{j=1}^{r} \text{area}(S_j) < \frac{\varepsilon}{b-a} \sum_{i=1}^{r} (t_i - t_{i-1}) = \varepsilon.
\]

Done.

**Remark.** It is not true that the image of any continuous map \( \alpha : [a, b] \to \mathbb{R}^2 \) has content zero. In fact, there are so called Peano curves which fill out an entire closed square \( [a_1, b_1] \times [a_2, b_2] \).

**Theorem 3** Let \( f \) be a bounded function on \( R \) which is continuous except on a subset \( Y \) of content zero. Then \( f \) is integrable on \( R \).

**Proof.** Let \( M > 0 \) be such that \( |f(x)| \leq M \), for all \( x \in R \). Since \( Y \) has content zero, we can find closed subrectangular boxes \( S_1, \ldots, S_m \) of \( R \) such that

(i) \( Y \subseteq \bigcup_{i=1}^{m} S_i \), and

(ii) \( \sum_{i=1}^{m} \text{vol}(S_i) < \frac{\varepsilon}{4M} \).

Extend \( \{S_1, \ldots, S_m\} \) to a partition \( P = \{S_1, \ldots, S_r\}, m < r \), of \( R \). Applying the small span theorem, we may suppose that \( S_{m+1} \ldots, S_r \) are so chosen that (for each \( i \geq m+1 \)) \( \text{span}_f(S_i) < \frac{\varepsilon}{2 \text{vol}(R)} \). (We can apply this theorem because \( f \) is continuous outside the union of \( S_1, \ldots, S_m \).) So we have

\[
U(f, P) - L(f, P) \leq 2M \sum_{i=1}^{m} \text{vol}(S_i) + \sum_{i=m+1}^{r} \text{span}_f(S_i) \text{vol}(S_i)
\]

\[
< (2M) \left( \frac{\varepsilon}{4M} \right) + \frac{\varepsilon}{2 \text{vol}(R)} \sum_{i=m+1}^{r} \text{vol}(S_i).
\]

But the right hand side is \( \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), because \( \sum_{i=m+1}^{r} \text{vol}(S_i) \leq \text{vol}(R) \). Done.

**Example.** Let \( R = [0, 1] \times [0, 1] \) be the unit square in \( \mathbb{R}^2 \), and \( f : R \to \mathbb{R} \) the function defined by \( f(x, y) = x + y \) if \( x \leq y \) and \( x - y \) if \( x > y \). Show that \( f \) is integrable on \( R \).

Let \( D = \{(x, x) \mid 0 \leq x \leq 1\} \) be the “diagonal” in \( R \). Then \( D \) has content zero as it is the graph of the continuous function \( \varphi(x) = x, 0 \leq x \leq 1 \). Moreover, \( f \) is discontinuous only
on $D$. So $f$ is continuous on $R \setminus D$ with $D$ of content zero, and consequently $f$ is integrable on $R$.

**Remark.** We can use this theorem to define the integral of a **continuous** function $f$ on any **bounded** set $B \subset \mathbb{R}^n$ if the boundary of $B$ has content zero. Indeed, in such a case, we may enclose $B$ in a closed rectangular box $R$ and define a function $\tilde{f}$ on $R$ by making it equal $f$ on $B$ and $0$ on $R \setminus B$. Then $\tilde{f}$ will be continuous on all of $R$ except for the boundary of $B$, which has content zero. So $\tilde{f}$ is integrable on $R$. Since $\tilde{f}$ is $0$ outside $B$, it is reasonable to set

$$\int_B f = \int_R \tilde{f}.$$  

In particular, we can define the **volume** of a bounded set $B$ with boundary of content zero by taking $f$ the constant function with value $1$:

$$\text{vol}(B) := \int_B 1 = \int_R \tilde{1}.$$  

### 4.4 Fubini’s theorem

So far we have been meticulous in figuring out when a given bounded function $f$ is integrable on $R$. But if $f$ is integrable, we have developed no method whatsoever to actually find a way to integrate it except in the really easy case of a step function. We propose to ameliorate the situation now by describing a very reasonable and computationally helpful result. We will state it in the plane, but there is a natural analog in higher dimensions as well. In any case, many of the intricacies of multiple integration are present already for $n = 2$, and it is a wise idea to understand this case completely at first.

**Theorem 4** (Fubini) Let $f$ be a bounded, integrable function on $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$. For $x$ in $[a_1, b_1]$, put $A(x) = \int_{a_2}^{b_2} f(x, y) \, dy$ and assume the following

(i) $A(x)$ exists for each $x \in [a_1, b_1]$, i.e., the function $y \mapsto f(x, y)$ is integrable on $[a_2, b_2]$ for any fixed $x$ in $[a_1, b_1]$;

(ii) $A(x)$ is integrable on $[a_1, b_1]$.

Then

$$\iint_R f(x, y) \, dxdy = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} f(x, y) \, dy \right) \, dx.$$  

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In other words, once the hypotheses (i) and (ii) are satisfied, we can compute \( \int_R f \) by performing two 1-dimensional integrals in order. One cannot always reverse the order of integration, however, and if one wants to integrate over \( x \) first, one needs to assume the obvious analog of the conditions (i), (ii).

**Proof.** Let \( P_1 = \{B_i \mid 1 \leq i \leq \ell\} \) (resp. \( P_2 = \{C_j \mid 1 \leq j \leq m\} \) be a partition of \([a_1, b_1]\) (resp. \([a_2, b_2]\)), with \( B_i, C_j \) closed intervals in \( \mathbb{R} \). Then \( P = P_1 \times P_2 = \{B_i \times C_j\} \) is a partition of \( R \). By hypothesis (i), we have

\[
L(f_x, P_2) \leq A(x) \leq U(f_x, P_2),
\]

where \( f_x \) is the one-dimensional function \( y \mapsto f(x, y) \). Then applying hypothesis (ii), we get

\[
L(L(f_x, P_2), P_1) \leq \int_{a_1}^{b_1} A(x) \, dx \leq U(U(f_x, P_2), P_1).
\]

But we have

\[
L(L(f_x, P_2), P_1) = \sum_{i=1}^{\ell} \text{length}(B_i) \inf(L(f_x, P_2)(B_i))
\]

\[
= \sum_{i=1}^{\ell} \text{length}(B_i) \inf_{x \in B_i} \sum_{j=1}^{m} \text{length}(C_j) \inf f(x, C_j)
\]

\[
\geq \sum_{i=1}^{\ell} \text{length}(B_i) \sum_{j=1}^{m} \text{length}(C_j) \inf(f(B_i \times C_j)) = L(f, P).
\]

We have used here computation rules such as \( \inf\{\lambda x \mid x \in S\} = \lambda \inf(S) \) for \( \lambda > 0 \) and \( \inf\{f(x) + g(x) \mid x \in S\} \geq \inf f(S) + \inf g(S) \) which are easy to verify. A similar estimate holds for the upper sum. Hence \( L(f, P) \leq \int_{a_1}^{b_1} A(x) \, dx \leq U(f, P) \). Given any partition \( Q \) of \( R \), we can find a partition \( P \) of the form \( P_1 \times P_2 \) which refines \( Q \). Thus \( L(f, Q) \leq \int_{a_1}^{b_1} A(x) \, dx \leq U(f, Q) \), for every partition \( Q \) of \( R \). Then by the uniqueness of \( \int_R f \), which exists because \( f \) is integrable, \( \int_{a_1}^{b_1} A(x) \, dx \) is forced to be \( \int_R f \). Done.

**Remark** The reason we denote \( \int_{a_2}^{b_2} f(x, y) \, dy \) by \( A(x) \) is the following. The double integral \( \iint_R f(x, y) \, dx \, dy \) is the volume subtended by the graph \( \Gamma = \{(x, y, f(x, y)) \in \mathbb{R}^3\} \) over the rectangle \( R \). (Note that \( \Gamma \) is a “surface” since \( f \) is a function of two variables.) When we fix \( x \) at the same point \( x_0 \) in \([a_1, b_1]\), the intersection of the plane \( \{x = x_0\} \) with \( \Gamma \) in \( \mathbb{R}^3 \) is a curve, which is none other than the graph \( \Gamma_{x_0} \) of \( f_{x_0} \) in the \((y, z)\)-plane shifted to \( x = x_0 \). The area under \( \Gamma_{x_0} \) over the interval \([a_2, b_2]\) is just \( \int_{a_2}^{b_2} f_{x_0}(y) \, dy \); whence the name \( A(x_0) \).
Note also that as $x_0$ goes from $a_1$ to $b_1$, the whole volume is swept by the slice of area $A(x_0)$.

A natural question to ask at this point is whether the hypotheses (i), (ii) of Fubini’s theorem are satisfied by many functions. The answer is yes, and the prime examples are continuous functions.

**Theorem 5** Let $f$ be a continuous function on $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$. Then $\int_R f$ can be computed as an iterated integral in either order. To be precise, we have

$$\int\int_R f(x, y) \, dxdy = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(x, y) \, dy \right] dx = \int_{a_2}^{b_2} \left[ \int_{a_1}^{b_1} f(x, y) \, dx \right] dy.$$

**Proof.** Since $f$ is continuous on (the compact set) $R$, it is certainly bounded. Let $M > 0$ be such that $|f(x, y)| \leq M$. We have also seen that it is integrable. For each $x$, the function $y \mapsto f(x, y)$ is integrable on $[a_2, b_2]$ because of continuity on $[a_2, b_2]$. So we get hypothesis (i) of Fubini. To get hypothesis (ii), it suffices to show that $A(x) = \int_{a_2}^{b_2} f(x, y) \, dy$ is continuous in $x$. For $h$ small, we have

$$|A(x + h) - A(x)| = \left| \int_{a_2}^{b_2} (f(x + h, y) - f(x, y)) \, dy \right| \leq \int_{a_2}^{b_2} |f(x + h, y) - f(x, y)| \, dy.$$

By the small span theorem we can find a partition $\{S_j\}$ of $R$ with $\text{span}_f(S_j) < \epsilon/(b_2 - a_2)$. If $h$ is small enough so that $(x + h, y)$ and $(x, y)$ lie in the same box for all $y$ (which we can achieve since $x$ is fixed and there are only finitely many boxes) we have $|f(x + h, y) - f(x, y)| < \text{span}_f(S_j) < \epsilon/(b_2 - a_2)$ where $S_j$ is a box containing both points. Note that this argument also works if $(x, y)$ lies on the vertical boundary between two boxes: for positive $h$ we land in one box and for negative $h$ in the other. Hence

$$\int_{a_2}^{b_2} |f(x + h, y) - f(x, y)| \, dy < \epsilon$$

for $h$ sufficiently small. This shows that $A(x)$ is continuous and hence integrable on $[a_1, b_1]$. We have now verified both hypotheses of Fubini, and hence

$$\int\int_R f(x, y) \, dxdy = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(x, y) \, dy \right] dx.$$

To prove that $\int_R f$ is also computable using the iteration in reverse order, all we have to do is note that by a symmetrical argument, the integral $\int_{a_1}^{b_1} f(x, y) \, dx$ makes sense and is
continuous in $y$, hence integrable on $[a_2, b_2]$. The Fubini argument then goes through. Done.

**Remark.** We will note the following extension of the theorem above without proof.

Let $f$ be a continuous function on a closed rectangular box $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then the integral of $f$ over $R$ is computable as an iterated integral

$$\int_{a_1}^{b_1} \left[ \cdots \left( \int_{a_n}^{b_n} f(x_1, \ldots, x_n) \, dx_n \right) \, dx_{n-1} \right] \cdots \, dx_1.$$ 

Moreover, we can compute this in any order we want, e.g., integrate over $x_2$ first, then over $x_5$, then over $x_1$, etc. Note that there are $n!$ possible ways here of permuting the order of integration.

### 4.5 Integration over special regions

Right before section 4.4 we’ve already defined the integration over a bounded region $B$ provided the boundary of $B$ has content zero. A particular example of bounded sets are compact sets (recall: compact= closed and bounded). Hence we have

**Theorem 6** Let $Z$ be a compact subset of $\mathbb{R}^n$ such that the boundary of $Z$ has content zero. Then any function $f$ on $Z$ which is continuous on $Z$ is integrable over $Z$.

Now we will analyze the simplest cases of this phenomenon in $\mathbb{R}^2$.

**Definition.** A region of type I in $\mathbb{R}^2$ is a set of the form \( \{a \leq x \leq b, \, \varphi_1(x) \leq y \leq \varphi_2(x)\} \), where $\varphi_1, \varphi_2$ are continuous functions on $[a, b]$.

A region of type II in $\mathbb{R}^2$ is a set of the form \( \{c \leq y \leq d, \, \psi_1(y) \leq x \leq \psi_2(y)\} \), where $\psi_1, \psi_2$ are continuous functions on $[c, d]$.

A region of type III in $\mathbb{R}^2$ is a subset which is simultaneously of type I and type II.

**Remark.** Note that a circular region is of type III. Note also that a set of type I,II or III is compact as it is closed an bounded.

**Theorem 7** Let $f$ be a continuous function on a subset $S$ of $\mathbb{R}^2$. 
(a) Suppose \( S \) is a region of type I defined by \( a \leq x \leq b \), \( \varphi_1(x) \leq y \leq \varphi_2(x) \), with \( \varphi_1, \varphi_2 \) continuous. Then \( f \) is integrable on \( S \) and

\[
\int_S f = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, dy \right) \, dx.
\]

(b) Suppose \( S \) is a region of type II defined by \( c \leq y \leq d \), \( \psi_1(y) \leq x \leq \psi_2(y) \), with \( \psi_1, \psi_2 \) continuous. Then \( f \) is integrable on \( S \) and

\[
\int_S f = \int_c^d \left( \int_{\psi_1(x)}^{\psi_2(x)} f(x, y) \, dx \right) \, dy.
\]

Proof. We will prove (a) and leave the symmetrical case (b) to the reader.

(a) Let \( R = [a, b] \times [c, d] \), where \( c, d \) are chosen so that \( R \) contains \( S \). Define \( \tilde{f} \) on \( R \) as above (by extension of \( f \) by zero outside \( S \)). By the Proposition of §5.3, we know that the graphs of \( \varphi_1 \) and \( \varphi_2 \) are of content zero, since \( \varphi_1, \varphi_2 \) are continuous. Thus the main theorem of §5.3 implies that \( \tilde{f} \) is integrable on \( R \) as its set of discontinuities is contained in the boundary \( \partial S \) of \( S \). It remains to prove that \( \int_S f = \int_R \tilde{f} \) is given by the iterated integral

\[
\int^b_a \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, dy \right) \, dx.
\]

For each \( x \in (a, b) \), the integral \( \int_c^d \tilde{f}(x, y) \, dy \) exists as the set of discontinuities in \( [c, d] \) has at most two points. Moreover, the function \( x \mapsto \int_c^d \tilde{f}(x, y) \, dy \) is continuous, except perhaps at \( a \) or \( b \), hence integrable on \( [a, b] \). Hence Fubini’s theorem applies in this context and gives

\[
\int_R f = \int^b_a \left( \int_c^d \tilde{f}(x, y) \, dy \right) \, dx.
\]

Since the inside integral (over \( y \)) is none other than \( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, dy \), the assertion of the theorem follows. Done.

4.6 Examples

(1) Compute \( \int_R f \), where \( R \) is the closed rectangle \([-1, 1] \times [2, 3] \) and \( f \) the function \( (x, y) \mapsto x^2 y - x \cos \pi y \). Since \( f \) is continuous on \( R \), we may apply Fubini’s theorem and compute \( \int_R I \) as the iterated integral

\[
I = \int_{-1}^1 \left( \int_2^3 (x^2 y - x \cos \pi y) \, dy \right) \, dx.
\]
Recall that in \( \int_2^3 (x^2 y - x \cos \pi y) \, dy \), \( x \) is treated like a constant, hence equals

\[
x^2 \int_2^3 y \, dy - x \int_2^3 \cos \pi y \, dy = x^2 \left( \frac{3^2}{2} - \frac{2^2}{2} \right) - x \left( \frac{\sin \pi y}{\pi} \right) \bigg|_2^3 = \frac{5}{2} x^2.
\]

\[\Rightarrow I = \frac{5}{2} \int_{-1}^1 x^2 \, dx = \frac{5}{2} \left( \frac{x^3}{3} \right) \bigg|_{-1}^1 = \frac{5}{3}.\]

We could also have computed it in the opposite order to get

\[
I = \int_2^3 \left[ \int_{-1}^1 (x^2 y - x \cos \pi y) \, dx \right] \, dy = \int_2^3 \left( y \left( \frac{x^3}{3} \right) \right) \bigg|_{-1}^1 - \cos \pi y \left( \frac{x^2}{2} \right) \bigg|_{-1}^1 \, dy
\]

\[
= \int_2^3 \left( \frac{2y}{3} \right) \, dy = \frac{y^2}{3} \bigg|_2^3 = \frac{5}{3}.
\]

(2) Find the volume of the tetrahedron \( T \) in \( \mathbb{R}^3 \) bounded by the planes \( x = 0, y = 0, z = 0 \) and \( x - y - z = -1 \).

Note first that the base of \( T \) is a triangle \( \triangle \) defined by \(-1 \leq x \leq 0, 0 \leq y \leq x + 1\). Given any \((x, y)\) in \( \triangle \), the height of \( T \) above it is simply given by \( z = x - y + 1 \). Hence we get by Theorem 7

\[
\text{vol}(T) = \int \int_{\triangle} (x - y + 1) \, dxdy = \int_0^1 \left( \int_{-1}^{x+1} (x - y + 1) \, dy \right) \, dx
\]

\[
= \int_{-1}^0 \left( xy - \frac{y^2}{2} + y \right) \bigg|_0^{x+1} \, dx
\]

\[
= \int_{-1}^0 \frac{(x+1)^2}{2} \, dx = \int_0^1 \frac{u^2}{2} \, du = \frac{1}{6}.
\]

We could also have computed it in the opposite order to get

\[
I = \int_2^3 \left[ \int_{-1}^1 (x^2 y - x \cos \pi y) \, dx \right] \, dy
\]

\[
= \int_2^3 \left( y \left( \frac{x^3}{3} \right) \right) \bigg|_{-1}^1 - \cos \pi y \left( \frac{x^2}{2} \right) \bigg|_{-1}^1 \, dy
\]

\[
= \int_2^3 \left( \frac{2y}{3} \right) \, dy = \frac{y^2}{3} \bigg|_2^3 = \frac{5}{3}.
\]

(3) Fix \( a, b > 0 \), and consider the region \( S \) inside the ellipse defined by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) in \( \mathbb{R}^2 \). Compute \( I = \int \int_S \sqrt{a^2 - x^2} \, dxdy \).

Note that \( S \) is a region of type I as we may write it as

\[
\left\{ -a \leq x \leq a, -b \sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b \sqrt{1 - \frac{x^2}{a^2}} \right\}.
\]
Since the function \((x, y) \mapsto \sqrt{a^2 - x^2}\) is continuous, we can apply the main theorem of §5.5. We obtain

\[
I = \int_{-a}^{a} \sqrt{a^2 - x^2} \left( \int_{-b}^{b} \sqrt{1 - \frac{x^2}{a^2}} \, dy \right) \, dx
\]

\[
= \frac{2b}{a} \int_{-a}^{a} (a^2 - x^2) \, dx = \frac{2b}{a} \left( a^3 x - \frac{x^3}{3} \right) \bigg|_{-a}^{a}
\]

\[
= 4a^2 b - \frac{4a^2 b}{3} = \frac{8a^2 b}{3}.
\]

4.7 Applications

Let \(S\) be a thin plate in \(\mathbb{R}^2\) with matter distributed with density \(f(x, y)\) (= mass/unit area). The mass of \(S\) is given by

\[
m(S) = \iint_S f(x, y) \, dxdy.
\]

The average density is

\[
\frac{m(S)}{\text{area}} = \frac{\iint_S f(x, y) \, dxdy}{\iint_S dxdy}.
\]

The center of mass of \(S\) is given by \(\bar{z} = (\bar{x}, \bar{y})\), where

\[
\bar{x} = \frac{1}{m(S)} \iint_S x f(x, y) \, dxdy
\]

and

\[
\bar{y} = \frac{1}{m(S)} \iint_S y f(x, y) \, dxdy.
\]

When the density is constant, the center of mass is called the centroid of \(S\).

Suppose \(L\) is a fixed line. For any point \((x, y)\) on \(S\), let \(\delta = \delta(x, y)\) denote the (perpendicular) distance from \((x, y)\) to \(L\). The moment of inertia about \(L\) is given by

\[
I_L = \iint_S \delta^2(x, y) f(x, y) \, dxdy.
\]

When \(L\) is the \(x\)-axis (resp. \(y\)-axis), it is customary to write \(I_x\) (resp. \(I_y\)).

Note that the center of mass is a linear invariant, while the moment of inertia is quadratic.

An interesting use of the centroid occurs in the computation of volumes of revolutions. To be precise we have the following
Theorem 8 (Pappus) Let $S$ be a region of type I, i.e., given as $\{a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with $\varphi_1, \varphi_2$ continuous. Suppose that $\min_x \varphi_1(x) > 0$, so that $S$ lies above the $x$-axis. Denote by $V$ the volume of the solid $M$ obtained by revolving $S$ about the $x$-axis, and by $\bar{z} = (\bar{x}, \bar{y})$ the centroid of $S$. Then

$$V = 2\pi \bar{y} a(S),$$

where $a(S)$ is the area of $S$.

Proof. Let $R_i$ be the solid obtained by revolving the area $\{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, 0 \leq y \leq \varphi_i(x)\}$ about the $x$-axis. Then we can describe $R_i$ as a type I region in $\mathbb{R}^3$ (extending our definition of type I to higher dimensions in a straightforward way), namely

$$R_i = \{(x, y, z) \in \mathbb{R}^3 | a \leq x \leq b, -\varphi_i(x) \leq y \leq \varphi_i(x), -\sqrt{\varphi_i(x)^2 - y^2} \leq z \leq \sqrt{\varphi_i(x)^2 - y^2}\}.$$  

So we find

$$\text{vol}(R_i) = \int_{R_i} 1 = \int_{a}^{b} dx \int_{-\varphi_i(x)}^{\varphi_i(x)} dy \int_{-\sqrt{\varphi_i(x)^2 - y^2}}^{\sqrt{\varphi_i(x)^2 - y^2}} dz = \int_{a}^{b} \pi \varphi_i(x)^2 dx.$$  

But clearly, $M = R_2 \setminus R_1$ and $V = \text{vol}(R_2) - \text{vol}(R_1)$. So we have

$$V = \pi \int_{a}^{b} [\varphi_2(x)^2 - \varphi_1(x)^2] dx.$$  

On the other hand, we have by the definition of the centroid,

$$\bar{y} = \frac{1}{a(S)} \iint_S y dxdy.$$  

Since $y$ is continuous and $S$ a region of type I, we have

$$\bar{y} = \frac{1}{a(S)} \int_{a}^{b} \left( \int_{\varphi_1(x)}^{\varphi_2(x)} y dy \right) dx = \frac{1}{a(S)} \int_{a}^{b} \frac{1}{2} [\varphi_2(x)^2 - \varphi_1(x)^2] dx.$$  

The theorem now follows immediately. Done.

Examples (1) Let $S$ be the semi-circular region $\{-1 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2}\}$. Compute the centroid of $S$.  

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Since $S$ is of type I, we have

$$a(S) = \iint_S dx \, dy = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy$$

$$= \int_{-1}^{1} \sqrt{1-x^2} \, dx = 2 \int_0^{1} \sqrt{1-x^2} \, dx.$$ 

Put $x = \sin t$, $0 \leq t \leq \frac{\pi}{2}$. Then $dx = \cos t \, dt$ and $\sqrt{1-x^2} = \cos t$. So we get

$$a(S) = 2 \int_0^{\pi/2} \cos^2 t \, dt = 2 \int_0^{\pi/2} \left( \frac{1 + \cos t}{2} \right) \, dt$$

$$= 2 \left[ \frac{\pi}{4} + \frac{\sin 2t}{4} \right]_0^{\pi/2} = \frac{\pi}{2}.$$ 

Of course, we could have directly reasoned by geometry that the area of a semi-circular region of radius 1 is $\frac{\pi}{2}$.

Let $\bar{z} = (\bar{x}, \bar{y})$ be the centroid.

$$\bar{x} = \frac{1}{a(S)} \int_S x \, dx \, dy = \frac{2}{\pi} \int_{-1}^{1} \left( \int_0^{\sqrt{1-x^2}} y \, dy \right) x \, dx$$

$$= \frac{2}{\pi} \int_{-1}^{1} x \sqrt{1-x^2} \, dx = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \sin t \cos^2 t \, dt = 0,$$

since the integrand is an odd function.

Again, the fact that $\bar{x} = 0$ can be directly seen by geometry. The key thing is to compute $\bar{y}$. We have

$$\bar{y} = \frac{2}{\pi} \int_{-1}^{1} x \left( \int_0^{\sqrt{1-x^2}} y \, dy \right) \, dx = \frac{1}{\pi} \int_{-1}^{1} (1-x^2) \, dx$$

$$= \frac{1}{\pi} \left( x - \frac{x^3}{3} \right)_{-1}^{1} = \frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi}.$$ 

So the centroid of $S$ is $(0, \frac{4}{3\pi})$.

(2) Find the volume $V$ of the torus $T$ obtained by revolving about the $x$-axis a circular region $S$ of radius $r$ (lying above the $x$-axis).
The area $a(S)$ is $\pi r^2$, and the centroid $(\bar{x}, \bar{y})$ is located at the center of $S$ (easy check!). Let $b$ be the distance from the center of $S$ to the $x$-axis. Then by Pappus’ theorem,

$$V = 2 \pi \bar{y} a(S) = 2 \pi b (\pi r^2) = 2 \pi^2 r^2 b.$$