In the following problems, unless otherwise specified, $G$ is a connected compact Lie group, and $\int_G$ is the normalized Haar integral on $G$.

1. Consider the Lie group $\text{SL}(2, \mathbb{R})$. Show that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is contained in the image of the exponential map. Show that, however, the exponential map is not surjective.

2. Show that the center $Z(G)$ of $G$ is the intersection of all maximal tori of $G$.

3. (a) Let $G = U(n)$. Let $T \subset G$ be the subgroup of diagonal matrices. Show that $T$ is a maximal torus of $G$. Show that the Weyl group of $G$ is isomorphic to the group of permutations on $n$ letters.
   
   (b) (Optional) Find a maximal torus for $\text{SO}(n)$ and calculate its Weyl group.

4. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbb{F}$. The Killing form on $\mathfrak{g}$ is defined as $B(X, Y) = \text{tr}(\text{ad} X \cdot \text{ad} Y)$.

   (1) If $\mathfrak{g} = \text{Lie } G$ for some Lie group $G$, show that $B$ is $G$-invariant.

   (2) If $G$ is a compact Lie group, show that $B$ is negative semi-definite. i.e. $B(X, X) \leq 0$.

   A Lie algebra is called semisimple if $B$ is non-degenerate. A (connected) closed subgroup of $\text{GL}(n, \mathbb{R})$ is called semisimple if its Lie algebra is semisimple.

   (3) Show that the following conditions for a connected compact group are equivalent.

   (i) $G$ is semisimple; (ii) $B$ is negative definite; (iii) $Z(G)$ is finite.

   (4) (Optional) If $G$ is a connected Lie group such that $B$ is negative definite, then $G$ is compact.

5. In this problem, we study some geometry and topology of the (very important) manifold $G/T$. To study $G/T$, without loss of generality, we can assume that $G$ is semisimple.

   Let $W$ be the Weyl group of $G$ (w.r.t to $T$).

   (1) Let $H \subset \mathfrak{t}$ such that $\alpha(H) \neq 0$ for all roots $\alpha$. Show that the centralizer of $H$ in $G$ (under the adjoint representation) is $T$. We therefore obtain an embedding $G/T \to \mathfrak{g}$, $gT \mapsto \text{Ad}_g H$.

   (2) We define a smooth function $f : G/T \to \mathbb{R}$ as $f(gT) = B(gH, H)$. Show that $W \simeq N_G(T)/T \subset G/T$ are exactly the points with $df = 0$. (Note that the Killing form $B$ induces a $G$-equivariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, and therefore $G/T$ is realized as a coadjoint orbit. In particular, $G/T$ has a symplectic structure. The function $f$ can be considered as a moment map.)

   (3) Given $w \in W$, with a lifting $n_w \in N_G(T)$, show that the Hessian of $f$ at $wT := n_w T$ is non-degenerate. In addition, it has $2\ell(w)$ negative eigenvalues, where $\ell(w)$ is the number of roots $\alpha$ satisfying $\alpha(H) > 0$ but $w(\alpha)(H) < 0$.

   (4) (Optional) Using the Morse theory to conclude that

   $H^i(G/T, \mathbb{C}) \simeq \bigoplus_{w \in W, 2\ell(w) = i} \mathbb{C}$ if even, $i$ odd.

   In particular the Euler number $\chi(G/T) = |W|$.
(5) We construct a (left) action of $W$ of $G/T$ as follows: for $w \in W$, we define $w \cdot gT = gn_w^{-1}T$, where $n_w$ is a lift of $w$ to $N_G(T)$. This action induces a representation of $W$ on $H^*(G/T, \mathbb{C})$. Show that nontrivial elements $w \in W$ do not have fixed point under this action.

(6) Using (4), (5) and the Lefschetz fixed point theorem to conclude that $\text{tr}(w|H^*(G/T, \mathbb{C})) = 0$ if $w \neq 1$. Conclude that $H^*(G/T, \mathbb{C})$ is isomorphic to the regular representation of $W$.

(7) We construct a map $\theta : t^*_C \simeq \mathbb{X}^*(T) \otimes \mathbb{C} \to H^2(G/T, \mathbb{C})$ as follows. For every $\lambda \in \mathbb{X}^*(T)$, let $C_\lambda$ denote the corresponding 1-dimensional representation, and let $L_\lambda = G \times^x C_\lambda$ denote the complex line bundle on $G/T$. Then we define $\theta(\lambda)$ as the first Chern class $c_1(L_\lambda)$ of $L_\lambda$. Show that $c$ is $W$-equivariant, which induces an isomorphism $t^*_C \simeq H^2(G/T, \mathbb{C})$. 