1. (5.6.3) Prove directly from the definition that if $X_1, X_2, \cdots \in \{0, 1\}$ are exchangeable, then

$$P(X_1 = 1, X_2 = 1, \cdots, X_k = 1 | S_n = m) = \frac{{n-k \choose n-m}}{{n \choose m}}.$$ 

2. (6.2.6-ish, 6.5.2) A collection of otherwise indistinguishable black and white balls ($b$ black, $w$ white) are distributed between two urns. Compute the transition probabilities for the following Markov chains, and determine whether they are aperiodic and/or connected, and determine their stationary distributions: (a) At each time step, we pick a ball uniformly at random and move it to the other urn. (b) At each time step, we pick a ball from each urn uniformly at random, and swap. (c) At each time step we pick two distinct balls uniformly at random, and swap them.

3. (6.3.7-ish, 6.3.8) Let $X_n$ be a Markov chain with state space $\{0, 1, \ldots, N\}$ such that $X_n$ is also a martingale, and for any $x$, there is a positive probability of eventually reaching 0 or $N$. (a) Show that if we start at $x$, the probability of eventually reaching $N$ is $x/N$, and that of eventually reaching 0 is $1 - x/N$. (b) Show that the hypotheses apply to the chain for which at each time step, we generate i.i.d. uniform elements $Y_1, \ldots, Y_N$ of $\{0, 1, \ldots, N - 1\}$, and the new state is the number of these which are less than the current state.

4. (7.1.5-ish) Let $\phi(x) = 1/x - \lfloor 1/x \rfloor$ for $x \in (0, 1)$. (i) Show that $\phi$ preserves the distribution with density

$$\log 2^{-1} \frac{1}{1+x}.$$ 

(ii) If $a_0 = \lfloor 1/x \rfloor$, $a_1 = \lfloor 1/\phi(x) \rfloor$, $\ldots$, then $x$ has the continued fraction expansion $x = 1/(a_0 + 1/(a_1 + 1/\cdots))$. Assuming that $\phi$ is ergodic, what does this imply about the distribution of coefficients of continued fraction expansions?

5. (7.5.2) Let $\pi_n$ be a random permutation of $\{1, \ldots, n\}$, and let $J^k_n$ be the number of increasing subsequences of length $k$ of $\pi_n$. Compute the expected value of $J^k_n$, and use this to conclude that $\lim \sup_{n \to \infty} E(\ell(\pi_n)/\sqrt{n}) \leq e$, where $\ell(\pi_n)$ is the length of the longest increasing subsequence of $\pi_n$. 
