Math 144a Probability
Homework Set 1
Due Wednesday 2/4/15 in class
(or in the 144 box next to the department office)

1. (1.6.1) Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly convex function (i.e., $t\phi(x)+(1-t)\phi(y) > \phi(tx+(1-t)y)$ for all $0 < t < 1$, $x < y$), and that $X$ is a random variable such that $E(\phi(X)) = \phi(E(X))$. Show that $X - E(X)$ is almost surely 0.

2. (1.6.10-ish) Let $A_1, A_2, \ldots$, be a possibly infinite sequence of events, with union $A$. Show that
   \[ \sum_i \Pr(A_i) - \sum_{i<j} \Pr(A_i \cap A_j) \leq \Pr(A) \leq \sum_i \Pr(A_i). \]
   (Hint (per the book): write everything as an expectation.)

3. Let $U_1, \ldots, U_n$ be i.i.d. random variables uniformly distributed in $(0, 1)$. Show that the random variable $\max(U_1, \ldots, U_n)$ is also uniformly distributed in $(0, 1)$.

4. The standard normal distribution (denoted $N(0, 1)$) is the absolutely continuous probability distribution with density $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$. (a) Show that if $X, Y$ are independent standard normals, then so are $(X + Y)/\sqrt{2}$ and $(X - Y)/\sqrt{2}$. (b) More generally, show that if $X_1, \ldots, X_n$ are i.i.d. standard normals, then for any $n \times n$ matrix $A$ such that $AA^t = 1$, the variables $Y_i = \sum_j A_{ij}X_j$ are also i.i.d. standard normals. (c) Conclude that if $X_1, \ldots, X_n$ are i.i.d. standard normals and $\alpha_1, \ldots, \alpha_n$ are not all 0, then
   \[ \frac{\sum_i \alpha_i X_i}{\sqrt{\sum_i \alpha_i^2}} \]
   is standard normal.

5. (2.2.4) Let $X_1, X_2, \ldots$ be i.i.d. with $\Pr(X_i = (-1)^k k) = \frac{C}{k^2 \log k}$ for $k \geq 2$, where
   \[ C = \sum_{k \geq 2} \frac{1}{k^2 \log k}. \]
   Show that this distribution does not have an expectation, but that there still exists a finite constant $\mu$ such that
   \[ \frac{X_1 + \cdots + X_n}{n} \to \mu \]
   in probability. (Hint: Theorem 2.2.7 looks useful for this)

6. (2.4.3) Let $X_0 = (1, 0)$ and define $X_n \in \mathbb{R}^2$ inductively by taking $X_{n+1}$ to be uniformly distributed in the disc of radius $|X_n|$ around the origin. Find (and prove) the constant $c$ such that $n^{-1} \log |X_n| \to c$ a.s.