Homework set 1, Ma121b, Winter 2015
Due Thursday January 22, 2015 at the beginning of class

1. Since formal power series are themselves a ring, we may consider modules over the ring of formal power series, and in particular free modules and matrices.

   (a) As a first application of this, consider the following enumeration problem: How many binary sequences of length \( n \) do not contain a subsequence 00? Let \( F_0(x) \) be the generating function of legal binary sequences that end in 0, and let \( F_1(x) \) be the corresponding generating function of sequences ending in 1. Find a recurrence relating the coefficients of \( F_0 \) and \( F_1 \), express it in terms of the vector \( (F_0 \ F_1) \), and use that to solve for the generating functions.

   (b) More generally, suppose we are given a finite set \( \Sigma \) of bit strings, and wish to know the number of bit strings such that no substring is in \( \Sigma \). Prove that the generating function is a rational function of \( x \), and give a procedure for computing it.

   (c) A finite automaton is determined by the following data: a finite set \( A \) (the alphabet), a finite set \( S \) of states, and a transition function \( S \times A \to S \). In other words, each element of \( A \) determines a function from \( S \) to \( S \), and thus any string of elements of \( A \) determines such a function by composition. For each pair \( s_1, s_2 \in S \), consider the generating function enumerating strings on \( A \) such that the corresponding function takes \( s_1 \) to \( s_2 \). Prove that these generating functions are rational functions, and give a procedure for computing them.

2. Consider a weighted coin which has chance \( p \) of landing heads and \( 1-p \) of landing tails. We throw the coin \( n \) times (and assume the results are independent). Let the random variable \( X \) denote the number of times the coin came up heads, and let \( p_k := \Pr(X = k) \).

   (a) Calculate the generating function \( F_n(x) = \sum_{0 \leq k \leq n} p_k x^k \).

   (b) Use \( F_n \) to calculate the expectation and variance of \( X \).

   (c) Give an interpretation of the expectation \( \text{Exp}(e^{tX}) \) as a generating function.

   (d) Compute that generating function.

3. (a) Give a combinatorial proof that

   \[
   \prod_{0 \leq k} (1 + q^{2k+1}z) = \sum_{0 \leq k} q^{k^2} z^k / \prod_{1 \leq j \leq k}(1 - q^{2j})
   \]

   (b) Give a combinatorial proof that

   \[
   (1 + q + \ldots q^{n-1}) \binom{m+n-1}{n} = (1 + q + \ldots q^{m-1}) \binom{m+n-1}{m} q
   \]
using the interpretation of $q$-binomial coefficients as enumerating partitions with a bounded number of bounded parts.

(c) As part (b), but now using the interpretation via counting subspaces of $\mathbb{F}_q$. 