(1) Recall that an extension of a group $G$ by a group $A$ just means a short exact sequence of groups $1 \to A \to \tilde{G} \to G \to 1$, and that two extensions are considered equivalent if you can write down the obvious commutative diagram with the outside maps given by the identity maps on $G$ and $A$. Now suppose that $A$ is abelian. Then, given an extension as above, we get an action of $G$ on $A$ by lifting elements of $G$ to $\tilde{G}$ and acting on $A$ by conjugation. (The abelianness of $A$ means that this action is independent of the lift.) Given an action of $G$ on an abelian group $A$, show that $H^2(G, A)$ classifies extensions of $G$ by $A$ where the induced action is the given one. (Hint: Given an extension, choose a lifting $F(g)$ of each element of $G$ up to $\tilde{G}$. Using associativity of multiplication in $\tilde{G}$, show that the expression $f(g, h) = F(g)F(h)F(gh)^{-1}$ gives a 2-cocycle and that modifying the lifting $F$ will modify $f$ by a coboundary – this gives a map from the set of extension classes to $H^2$. To go backwards, just use a 2-cocycle to write down a group law on the set $A \times G$ by inverting the above construction.)

(2) Show that the homology group $H_1(G, \mathbb{Z})$ is naturally isomorphic to the abelianization of $G$ as follows. If $\Lambda = \mathbb{Z}[G]$ is the group ring, we have a natural surjection $\epsilon : \Lambda \to \mathbb{Z}$ defined by $\epsilon(g) = 1$ for each $g$. The kernel of this map is called $I_G$, the augmentation ideal, and it is obviously generated by elements of the form $1 - g$.

(a) Show that for any $G$-module $A$ we have that the coinvariants $A_G = A/I_G A$.

(b) Use the fact that $H_1(G, \Lambda) = 0$ (by freeness) show that $H_1(G, \mathbb{Z}) = I_G/I_G^2$.

(c) Show directly that this last group is isomorphic to the abelianization of $G$ using the map $G \to I_G$ defined by $g \mapsto 1 - g$.

(3) In class we have always discussed the cohomology in the setting of a $G$-module, $A$, that is, an abelian group $A$ with an action of $G$. For the first cohomology group, $H^1(G, A)$, it is possible to dispense with the hypothesis that $A$ is abelian, by directly setting $H^1(G, A)$ to be crossed homomorphisms modulo “boundaries”

(a) Write this down carefully (by replacing the expression $\sigma + \partial a$ which expresses cohomologousness in the abelian case by $a\sigma(g)g(a)^{-1}$) and verify that it gives a functor from the category of groups with a $G$ action to the category of pointed sets. (This $H^1$ does not have a group structure, but it does have a “zero element.”)
(b) Check that given a short exact sequence of groups, you get a longer exact sequence of pointed sets

\[1 \to A^G \to A^G \to A'^G \to H^1(G, A') \to H^1(G, A) \to H^1(G, A'')\]

where exactness of a sequence of maps of pointed sets just means the usual thing, provided the kernel of a map is taken to mean the preimage of the distinguished point. (You will need to define the boundary map here, since it doesn’t follow from any general theory of derived functors.)

(c) In the above setting, if \(A'\) is contained in the center of \(A\), (and hence abelian, so that we have a notion of \(H^2(G, A')\)) show that we can define an appropriate boundary map from \(H^1(G, A'')\) to \(H^2(G, A')\) and add one more term to the above exact sequence (maintaining exactness.) (To define the boundary, try lifting a crossed homomorphism naively from \(A''\) to \(A\) and show that the failure of the lift to be crossed homomorphism defines a 2 cocycle with values in \(A'\).)