Math 120b: Homework Set 2
Due: Tuesday January 20 at 5PM
(All exercises are from Lang’s Algebra)

(1) Find the Galois group of \( x^3 - 10 \) over the fields \( \mathbb{Q}, \mathbb{Q}(\sqrt{2}), \) and \( \mathbb{Q}(\sqrt{3}) \).

(2) Find the Galois group of \( x^3 + x + 1 \) over \( \mathbb{Q} \).

(3) Let \( K \) be a separable extension of a field \( k \) such that the degree \([K : k]\) is a prime number \( p \). Let \( \theta \) in \( K \) be such that \( K = k(\theta) \) and let \( \theta_1, \ldots, \theta_p \) be the conjugates of \( \theta \) over \( k \) in some algebraic closure. If \( \theta_i \) is in \( k(\theta_j) \) for some \( i \neq j \), show that \( K \) is Galois over \( k \).

(4) Let \( k \) be a field whose characteristic is not 2. Let \( c \in k \), with \( c \notin k^2 \). Let \( F = k(\sqrt[4]{c}) \). Let \( \alpha = a + b\sqrt[4]{c} \) with \( a, b \in k \) and not both zero. Let \( E = F(\sqrt[4]{\alpha}) \). Prove that the following conditions are equivalent.
   \[ \begin{array}{l}
   \text{• } E \text{ is Galois over } k. \\
   \text{• } E = F(\sqrt[4]{\alpha'}) \text{ where } \alpha' = a - b\sqrt[4]{c}. \\
   \text{• } \text{Either } \alpha\alpha' = a^2 - cb^2 \in k^2 \text{ or } c\alpha\alpha' \in k^2. 
   \end{array} \]
   Show that when these conditions are satisfied, then \( E \) is cyclic over \( k \) of degree 4 if and only if \( c\alpha\alpha' \in k^2 \).

(5) Let \( f(x) = x^4 + ax^2 + b \) be an irreducible polynomial over \( \mathbb{Q} \), with roots \( \pm\alpha, \pm\beta \), and splitting field \( K \).
   \[ \begin{array}{l}
   \text{a) Show that the Galois group of } K/\mathbb{Q} \text{ is isomorphic to a subgroup of } D_8, \text{ the dihedral group with 8 elements, and thus isomorphic to either } \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \text{ or } D_8. \\
   \text{b) Show that the first case happens if and only if } \alpha - \beta \frac{a}{b} \text{ is rational.} \\
   \text{The second case happens if and only if } \alpha\beta \text{ or } \alpha^2 - \beta^2 \text{ is rational.} \\
   \text{(Although the case } \alpha^2 - \beta^2 \text{ does not actually occur – it would correspond to a subgroup of } D_8 \text{ isomorphic to } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \text{ but not acting transitively on the roots.)} \\
   \text{c) Find the splitting field } K \subset \mathbb{C} \text{ of the polynomial } x^4 - 4x^2 - 1. \\
   \text{Find the Galois group and work out the lattice of subfields of } K. 
   \end{array} \]

(6) Let \( K = \mathbb{C}(x) \) where \( x \) is transcendental over \( \mathbb{C} \), and let \( \zeta \) be a primitive cube root of unity in \( \mathbb{C} \). Let \( \sigma \) be the automorphism of \( K \) over \( \mathbb{C} \) such that \( \sigma(x) = \zeta x \). Let \( \tau \) be the automorphism of \( K \) over \( \mathbb{C} \) such that \( \tau(x) = x^{-1} \). Show that \( \sigma^3 = \tau^2 = 1 \) and \( \tau\sigma = \sigma^{-1}\tau \).
   Show that the group of automorphisms \( G \) generated by \( \sigma \) and \( \tau \) has order 6 and the subfield \( F \) of \( K \) fixed by \( G \) is the field \( \mathbb{C}(x^3 + x^{-3}) \).