Math 120b: Homework Set 1
Due: Monday January 12 at 5PM
(All exercises are from Lang’s Algebra)

(1) Let \( k \subset E \subset K \) be fields. Show that
\[
\text{tr.deg.}(K/k) = \text{tr.deg.}(K/E) + \text{tr.deg.}(E/k).
\]
If \( \{x_i\} \) is a transcendence base of \( E/k \) and \( \{y_j\} \) is a transcendence base of \( K/E \) then \( \{x_i, y_j\} \) is a transcendence base of \( K/k \).

(2) Let \( E = \mathbb{Q}(\alpha) \), where \( \alpha \) is a root of the equation \( \alpha^3 + \alpha^2 + \alpha + 2 = 0 \).
Express \( (\alpha^2 + \alpha + 1)(\alpha^2 + \alpha) \) and \( (\alpha - 1)^{-1} \) in the form \( a\alpha^2 + b\alpha + c \) with \( a, b, c \in \mathbb{Q} \).

(3) Show that \( \sqrt{2} + \sqrt{3} \) is algebraic over \( \mathbb{Q} \) of degree 4.

(4) Let \( \alpha \) be a real number with \( \alpha^4 = 5 \).
(a) Show that \( \mathbb{Q}(i\alpha^2) \) is normal over \( \mathbb{Q} \).
(b) Show that \( \mathbb{Q}(\alpha + i\alpha) \) is normal over \( \mathbb{Q}(i\alpha^2) \).
(c) Show that \( \mathbb{Q}(\alpha + i\alpha) \) is not normal over \( \mathbb{Q} \).

(5) Let \( K \) be a finite field with \( p^n \) elements. Show that every element of \( K \) has a unique \( p \)th root in \( K \).

(6) Let \( E = F(x) \) where \( x \) is transcendental over \( F \).
(a) Let \( K \neq F \) be a subfield of \( E \) which contains \( F \). Show (directly) that \( x \) is algebraic over \( K \).
(b) Let \( y = \frac{f(x)}{g(x)} \) be a rational function, with relatively prime polynomials \( f, g \) in \( F[x] \). Let \( n = \max(\deg(f), \deg(g)) \). Suppose \( n \geq 1 \). Prove that \( [F(x) : F(y)] = n \).

(7) Let \( k \) be a field of characteristic \( p \) and let \( t, u \) be algebraically independent over \( k \). Show that:
(a) \( k(t, u) \) has degree \( p^2 \) over \( k(t^p, u^p) \).
(b) There exist infinitely many extensions between \( k(t, u) \) and \( k(t^p, u^p) \).