1. (Problem 3.3.1 [10 points]) How many zeros does \( f(z) = z^{12} - 4z^8 + 9z^5 - 2z + 1 \) have in \( \{ z \mid |z| < 1 \} \)?

2. (Problem 3.3.2 [10 points]) How many zeros does \( f(z) = z^4 - 6z + 3 \) have in the annulus \( \{ z \mid 1 < |z| < 2 \} \)?

3. (Problem 3.10.1 [30 points]) The purpose of this problem is to show that if \( f(z) \) obeys \( f(z+1) = f(z) \) and \( f(z) = f(-z) \), then \( f(z) \) has a convergent expansion
\[
f(z) = \sum_{n=0}^{\infty} a_n (\cos(2\pi z))^n
\]
with
\[
|a_n| \leq C K e^{-Kn}
\]
for all \( K > 0 \).

(a) Prove that for \( w \neq \pm 1 \), \( \cos(2\pi z) = w \) has exactly two solutions in \( 0 \leq \text{Re} z < 1 \), and for \( w = 1 \) or \( w = -1 \), exactly one solution. (Hint: Look for \( \eta \) with \( \eta + \eta^{-1} = w \).)

(b) Prove there is a well-defined function, \( g \), on \( \mathbb{C} \) so that \( g(\cos(\pi z)) = f(z) \).

(c) Prove that \( w = \cos(\pi z) \) is locally an analytic bijection if \( w \neq \pm 1 \) and conclude \( g \) is analytic on \( \mathbb{C} \setminus \{ \pm 1 \} \).

(d) Prove that \( f \) only has even terms in its power series about \( z = 0 \) and about \( z = \frac{1}{2} \) and conclude \( g \) is an entire function.

(e) Conclude (1) and (2) hold.

4. (Problem 4.7.4 [30 points]) The purpose of this problem is to lead the reader through a rather remarkable application of Runge’s theorem, the existence of a sequence of polynomials, \( p_n \) (Note: \( p_n \) is not claimed to have degree \( n! \)) so that
\[
\lim_{n \to \infty} p_n(0) = 1 \quad \text{For any fixed } z \neq 0, \lim_{n \to \infty} p_n(z) = 0
\]
We’ll comment on the significance of these sequences below.

(a) Let (see Fig. 1)
\[
K_n = \{ z \mid |z| \leq n \} \setminus \{ z \mid \text{dist}(z, [0, n]) < n^{-1} \} \cup \{0\} \cup \left[ \frac{1}{n}, n \right]
\]
Prove that \( K_n \) is compact and \( \mathbb{C} \setminus K_n \) is connected.

(b) Let \( f_n \) be the continuous function on \( K_n \) with \( f_n(0) = 1 \) and \( f_n(z) = 0 \) for \( z \in K_n \setminus \{0\} \). Use Runge’s Second Theorem to find a polynomial, \( p_n \), with
\[
\sup_{K_n} |p_n(z) - f_n(z)| \leq \frac{1}{n}
\]
(c) Show that \( p_n \) obeys (3).
5. (Problem 4.8.1 [20 points]) Let $\gamma$ be a simple $C^1$ arc and $f$ a $C^1$ function on $\text{Ran}(\gamma)$ supported on $\gamma([\varepsilon, 1 - \varepsilon])$ for some $\varepsilon$. For $w \notin \text{Ran}(\gamma)$, define the contour integral (Cauchy transform of $f$),

$$C_f(w) = \frac{1}{2\pi i} \int \frac{f(z)}{w - z} \, dz$$

Fix $\varepsilon > 0$. For $z_0 \in \text{Ran}(\gamma)$, define $\Gamma^\pm(z_0) = \{w : w = z_0 \pm re^{i(\theta + \theta_0)} : 0 < r < \rho, |\theta| < \frac{\pi}{2} - \varepsilon\}$, where $\rho$ is picked so small that $\Gamma^\pm \cap \text{Ran}(\gamma) = \emptyset$, and $\theta_0(z)$ is defined so that $e^{-i\theta_0(z)}\gamma'(z_0)$ is a positive imaginary (normal and to the left; see Fig. 5).

(a) If $f(z_0) = 0$, prove that $C_f(w)$ can be extended to $\Gamma^+(z_0) \cup \Gamma^-(z_0) \cup \{z_0\}$ to be continuous at $z_0$ (i.e., $C_f(w)$ has equal nontangential limits at $z_0$).

(b) Using Proposition 4.8.3, prove the Cauchy Jump Formula that for any such $f$ and all $z_0$, we have

$$\lim_{w \to z_0 \atop w \in \Gamma^+(z_0)} C_f(z) - \lim_{w \to z_0 \atop w \in \Gamma^-(z_0)} C_f(z) = f(z_0) \quad (5)$$

6. (Problem 5.3.8 [20 points]) A Carathéodory function is an analytic function on $\mathbb{D}$ with $F(0) = 1$ and $\text{Re} F(z) \geq 0$ on $\mathbb{D}$. Suppose

$$F(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (6)$$
Prove (a theorem of Borel [106]) that for such functions, \(|c_n| \leq 2\). \((\text{Hint: Use the complex Poisson representation to prove for each } r \in (0, 1),\)

\[
c_n = 2r^{-n} \int e^{+in\theta} \text{Re } F(re^{i\theta}) \frac{d\theta}{2\pi}
\]

and take \(r \to 1\) after using \(\int \text{Re } F(re^{i\theta}) \frac{d\theta}{2\pi} = 1\).

7. (Problem 5.7.26 [30 points]) Gaussian sums are defined as

\[
g_n = \sum_{j=0}^{n-1} e^{2\pi ij^2/n}
\]

In this problem, use contour integrals to compute \(g_n\) and, as a bonus, Fresnel integrals and therefore Gaussian integrals!

(a) Let \(f(z) = e^{2\pi iz^2/n}/(e^{2\pi i} - 1)\). Let \(\Gamma_R\) be the rectangle with corners, \(\pm iR, \frac{1}{2}n \pm iR\) oriented counterclockwise. There is a pole of \(f\) at 0, and if \(n\) is even at \(n/2\), so a principal part is needed. Prove that \(\text{Res}(f; k) = (2\pi i)^{-1} e^{2\pi ik^2/n}\) and then that, for any \(R\),

\[
\frac{1}{2} g_n = \text{pv} \oint_{\Gamma_R} f(z) \, dz
\]

(b) Let \(H_R\) and \(V_R\) be the horizontal and vertical sides of the contour. Prove that as \(R \to \infty\), \(H_R = O(1/R)\) by first proving that for \(R\) large,

\[
|f(x + iR)| \leq 2e^{-4\pi x R/n} \quad |f(x - iR)| \leq 2e^{-4\pi (\frac{2}{n} - x)/n}
\]

(c) Prove that \(f(iy) + f(-iy) = -e^{-2\pi iy^2/n}\) and that \(f(\frac{n}{2} + iy) + f(\frac{n}{2} - iy) = i^{3n} e^{-2\pi iy^2/n}\), and then that

\[
V_R = i(1 + i^{3n}) \int_0^R e^{-2\pi iy^2/n} \, dy
\]

(d) Conclude that

\[
g_n = 2i(1 + i^{3n}) \sqrt{n} \lim_{R \to \infty} \int_0^R e^{-2\pi iy^2} \, dy
\]

(e) Compute \(g_4\) and thereby that

\[
\lim_{R \to \infty} \int_0^R e^{-2\pi iy^2} \, dy = \frac{1}{4} (1 - i)
\]

From this, compute the basic Gaussian integral.

(f) Prove that \(g_n = \frac{1}{2}(1 + i)(1 + i^{3n})\sqrt{n}\).