Ma/CS 6b
Class 27: Art Galleries and Politicians

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The Art Gallery Problem

**Problem.** We wish to place security cameras at a museum, such that they cover it completely.

- Every camera can cover $360^0$ and has no distance limitation.
- The shape of the museum is a polygon with $n$ vertices, and no “holes”.
- We wish to minimize the number of cameras used.
Simple Cases

• What happens if the museum is a convex polygon with \( n \) vertices?
  ◦ One camera is required, and we can place it anywhere.

• This museum is a polygon with about 45 vertices that requires 6 cameras.
Better Examples

- Can you come up with museums that require many cameras?
  - This museum requires $\frac{n}{4}$ cameras.
  - Can we do better?
  - This Museum requires $\frac{n}{3}$ cameras.
  - Can we do better?
Naïve Upper Bounds

- Obviously \( n \) cameras always suffice, since we can place a camera at every vertex.
- If fact, \( n/2 \) cameras always suffice, since we can place one at every other vertex.
- Can we do better?
Theorem. A museum with $n$ vertices can be covered using at most $\lceil n/3 \rceil$ cameras.

This is tight, since we saw a museum that requires $n/3$.

Proof.

We consider the museum as a graph: a vertex for every corner and an edge for every wall.

This is a graph with $n$ vertices and $n$ edges.
Claim. Every simple polygon with \( n \geq 3 \) vertices can be triangulated using straight edges.

Proof. By induction on \( n \).

- **Induction basis.** Obvious for 3 and 4 vertices.
- **Induction step.** Recall that the sum of interior degrees in an \( n \)-gon is \( \pi(n - 2) \).
- **By the pigeonhole principle**, there exists a vertex \( a \) with interior degree smaller than \( \pi \).
  - \( b, c \) – the neighbors of \( a \) along the boundary.
  - If the edge \( (b, c) \) does not intersect the boundary, we add it. It remains to triangulate a polygon with \( n - 1 \) vertices, which can be done by the hypothesis.
Concluding the Induction Step

- $a$ – a vertex with interior degree smaller than $\pi$.
- $b, c$ – the neighbors of $a$ along the boundary.
- It remains to consider the case where the edge $(b, c)$ intersects the boundary.
- Slide the edge $(b, c)$ towards $a$ until it contains a single additional vertex $z$ of the boundary.
- We can add the edge $(a, z)$ and apply the induction hypothesis on the two resulting parts.
Triangulating

- We **trian**gulate the museum (but not the outer face). That is, we keep adding diagonals until no more can be added.

- We obtain a graph with $n$ vertices, $2n - 3$ edges, and $n - 2$ triangles.
• **Claim.** The triangulated graph can be colored using three colors.

• **Proof.** By induction on $n$.
  
  ◦ **Induction basis.** Obvious for $n = 3$.
  
  ◦ **Induction step.** Pick a diagonal $(u, w)$ of the triangulation. We can cut the museum into two pieces by removing $(u, w)$.
  
  ◦ **By the hypothesis**, each part can be colored with three colors.

  ◦ We permute the colors of one of the parts so that $u, w$ have the same colors in both parts.
Completing the Proof

- We proved that every museum can be triangulated with straight edges, and that the vertices of the triangulation can be colored with three colors.
- Each triangle must have all three colors.
- By the pigeonhole principle, there exists a color with at most \( \lceil n/3 \rceil \) vertices.
- We place a camera at each of these vertices. These cover all of the triangles and thus the entire museum.
An Open Variant

- **Problem.** Suppose that instead of cameras, we have guards. Each guard patrols along one wall. What is the minimum number of guards required?
- The following example requires $n/4$ guards, but it is not known whether every museum can indeed be covered using $n/4$. 
The Friendship Theorem

- **Theorem.** We have a group of people with the special property that every pair of people have exactly one common friend. Then there must be a person who is everybody’s friend (a politician).
  - We refer to this property as the friendship property.
Theorem. Let $G = (V, E)$ be a graph with the property that every two vertices have exactly one common neighbor. Then there exists $v \in V$ that is adjacent to every vertex of $V \setminus v$ (a politician vertex).

- What happens when $|V| = 3$?
- What happens when $|V| = 4$? No such graph.
- What happens when $|V| = 5$?
Warm Up

- **Claim.** A graph that satisfies the friendship property does not contain $C_4$ as a subgraph.

- **Proof.**
  - In a $C_4$ there are two vertices with two common neighbors, **contradicting the friendship property.**
A Stronger Formulation

- A windmill graph (or a friendship graph) consists of several triangles with one common vertex.
- **Theorem.** Any graph that satisfies the friendship property is a windmill graph.
Proof Strategy

- **Theorem.** Any graph that satisfies the friendship property is a windmill graph.
  - It suffices to prove that there exists a *politician* vertex \( v \in V \).
  - Since the graph contains no \( C_4 \), any vertex \( u \in V \setminus v \) must be connected to \( v \) and to exactly one vertex of \( V \setminus \{v, u\} \).
• **Claim.** Let $G = (V, E)$ satisfy the friendship condition, and consider $v, u \in V$ such that $(v, u) \notin E$. Then $\deg v = \deg u$.

• **Proof.**
  ◦ Denote the neighbors of $u$ as $\{w_1, \ldots, w_k\}$ such that $w_2$ is the common neighbor of $u, u$.
  ◦ Let $w_1$ be the common neighbor of $u$ and $w_2$.
  ◦ For every $2 \leq i \leq k$, let $z_i$ be the common neighbor of $v$ with $w_i$. Since there is no $C_4$ in $G$, the $z_i$’s are all distinct.
  ◦ This implies $\deg v \geq \deg u$.
  ◦ By a symmetric argument, we also have $\deg u \geq \deg v$. 

[Diagram showing relationships between nodes and edges]
Proof by Contradiction

• Assume, for contradiction, that there exists $G = (V, E)$ that satisfies the friendship condition and contains no politician vertex.

• Claim. $G$ is $d$-regular for some $d$.
  ◦ Consider $u, v \in V$ such that $(u, v) \notin E$. By the previous claim, $d = \deg u = \deg v$.
  ◦ Let $w$ be the common neighbor of $u$ and $v$. Every other vertex is not connected to both $u$ and $v$, and by the previous claim has degree $d$.
  ◦ Since $w$ is not a politician, it is also not connected to some vertex, and thus also has degree $d$. 
The Number of Vertices in $G$

- We assumed that there exists $G = (V, E)$ that satisfies the friendship condition and contains no politician vertex.
  - We proved that such a $G$ is $d$-regular.
- Consider a vertex $v \in V$.
  - We have $|N(v)| = d$ and by the friendship property, every vertex of $N(v)$ is adjacent to exactly one other vertex of $N(v)$.
  - $v$ is the only common neighbor of every two vertices of $N(v)$.
  - Thus, $|V| = 1 + d + d(d - 2) = d^2 - d + 1$. 
More Properties of $G$

- We know that $G$ is $d$-regular and has $d^2 - d + 1$ vertices.
  - When $d = 2$, we have $G = C_3$ which is a windmill graph. We thus assume $d \geq 3$.
  - This implies that the diameter of $G$ is 2.
  - Let $A$ be the adjacency matrix of $A$. We have
    $$A_{ij}^2 = \begin{cases} d, & \text{if } i = j, \\ 1, & \text{otherwise}. \end{cases}$$
    - That is, $A^2 = 1_{n \times n} + (d - 1)I$. 
Eigenvalues of $A^2$

- $A^2 = 1_{n \times n} + (d - 1)I$.
- Recall that $\text{spec}(1_{n \times n}) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}$.
- Thus, $\text{spec}(A^2) = \begin{pmatrix} n + d - 1 & d - 1 \\ 1 & n-1 \end{pmatrix}$.
- Since $n = d^2 - d + 1$, we have $n + d - 1 = d^2$. 

*Eigenvalues again?!?*
Eigenvalues of $A$

$$\text{spec}(A^2) = \begin{pmatrix} d^2 & d - 1 \\ 1 & d^2 - d \end{pmatrix}.$$ 

- The eigenvalues of $M$ are $\lambda_1, \ldots, \lambda_n$ iff the eigenvalues of $M^2$ are $\lambda_1^2, \ldots, \lambda_n^2$.
- Since $G$ is $d$-regular, $A$ has eigenvalue $d$.
- Thus,

$$\text{spec}(G) = \begin{pmatrix} d & \sqrt{d - 1} & -\sqrt{d - 1} \\ 1 & m_2 & m_3 \end{pmatrix}.$$ 

- That is, $d + (m_2 - m_3)\sqrt{d - 1} = 0$. 
A Useful Lemma

- **Lemma.** If \( m \) is an integer and \( \sqrt{m} \) is rational, then \( \sqrt{m} \) is an integer.

- **Proof.**
  - Assume, **for contradiction**, that \( \sqrt{m} \) is not an integer. Than there exists \( \ell \in \mathbb{N} \) such that \( 0 < \sqrt{m} - \ell < 1 \).
  - Let \( n_0 \) be the smallest positive integer such that \( n_0 \sqrt{m} \) is an integer.
  - Set \( n_1 = n_0 (\sqrt{m} - \ell) < n_0 \).
  - Then \( n_1 \sqrt{m} = n_0 m - \ell n_0 \sqrt{m} \). This is an integer, **contradicting the minimality of** \( n_0 \).
Completing the Friendship Proof

- $\text{spec}(G) = \begin{pmatrix} d & \sqrt{d-1} & -\sqrt{d-1} \\ 1 & m_2 & m_3 \end{pmatrix}$. 

- That is $d + (m_2 - m_3)\sqrt{d-1} = 0$.

- **By the lemma**, $s = \sqrt{d-1}$ must be an integer.

- $s(m_3 - m_2) = d = s^2 + 1$.

- This implies that $s$ divides $s^2 + 1$, so $s = 1$ and $d = 2$.

- **Contradicting our assumption that** $d \geq 3$. 


An Open Variant

• **Alternative formulation of the friendship property.** For every $u, v \in V$, there is a single path of length two between $u$ and $v$.

• **Conjecture.** For $\ell \geq 3$, no graph has the property that between every two vertices there is a single path of length $\ell$.
  ◦ This was proved for $3 \leq \ell \leq 33$, but remains open for general $\ell$. 

Now you know quite a bit about discrete math!