Recall: The Spectrum of a Graph

- Consider a graph $G = (V, E)$ and let $A$ be the adjacency matrix of $G$.
  - The **eigenvalues** of $G$ are the eigenvalues of $A$.
  - The **characteristic polynomial** $\phi(G; \lambda)$ is the characteristic polynomial of $A$.
  - The **spectrum** of $G$ is
    \[
    \text{spec}(G) = \left( \frac{\lambda_1, \ldots, \lambda_t}{m_1, \ldots, m_t} \right),
    \]
    where $\lambda_1, \ldots, \lambda_t$ are the eigenvalues of $A$ and $m_i$ is the multiplicity of $\lambda_i$. 

Example: Spectrum

\[ A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]

\[ \det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 & 0 & -1 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ -1 & 0 & -1 & \lambda \end{pmatrix} \]

\[ = \lambda^2(\lambda - 2)(\lambda + 2). \]

\[ \text{spec}(C_4) = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix} \]

Slight Change of Notation

- Instead of multiplicities, let \( \lambda_1, \ldots, \lambda_n \) be the not necessarily distinct eigenvalues of \( n \).

- For example, if the spectrum is \( \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \), we write \( \lambda_1 = \lambda_2 = 2 \) and \( \lambda_3 = \lambda_4 = -1 \) (instead of \( \lambda_1 = 2, m_1 = 2, \lambda_2 = -1, m_2 = 2 \)).
Recall: The Spectral Theorem

- **Theorem.** Any real symmetric $n \times n$ matrix has **real eigenvalues** and $n$ **orthonormal eigenvectors**.
  - By definition, any adjacency matrix $A$ is symmetric and real.
  - The algebraic and geometric multiplicities are the same in this case.
  - We have $\phi(A; \lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$.
  - The multiplicity of an eigenvalue $\lambda$ is $n - \text{rank}(\lambda I - A)$.

More Examples

- We already derived the following:
  - $\text{spec}(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$
  - $\text{spec}(K_{n,m}) = \begin{pmatrix} 0 & \sqrt{mn} & -\sqrt{mn} \\ m+n-2 & 1 & 1 \end{pmatrix}$

- Our next goal is to study **regular graphs**.
  - Can you come up with an eigenvector of any regular graph with $n$ vertices?
Eigenvalues of Regular graphs

• If $A$ is the adjacency matrix of a $d$-regular graph, then any row of $A$ contains exactly $d$ 1’s.
  ◦ Thus, the vector $1_n = (1, 1, ..., 1)$ is an eigenvector of $A$ with eigenvalue $d$.

• **Theorem.** Let $G$ be a connected graph. The eigenvalue of $G$ of largest absolute value is the maximum degree if and only if $G$ is regular.

**Proof**

• $A$ – $n \times n$ adjacency matrix of a graph $G$.
• $\Delta(G)$ – the maximum degree of $G$.
• $x = (x_1, ..., x_n)$ – eigenvector of eigenvalue $\lambda$ of largest absolute value.
• $x_j = \max_i |x_i|$.

\[
|\lambda| |x_j| = |(Ax)_j| = \left| \sum_{v_i \in N(v_j)} x_i \right| \\
\leq \deg v_j |x_j| \leq \Delta(G) |x_j|,
\]

• So $|\lambda| \leq \Delta(G)$.
• \(\Delta(G)\) – the maximum degree of \(G\).

• We proved that the absolute value of any eigenvalue of \(A\) is at most \(\Delta(G)\), using

\[
|\lambda| |x_j| = |(Ax)_j| = \left| \sum_{v_i \in N(v_j)} x_i \right| \leq \deg v_j |x_j| \leq \Delta(G) |x_j|.
\]

• For equality to hold, we need
  \(\circ \deg v_j = \Delta(G)\).
  \(\circ x_i = x_j\) for each \(v_i \in N(v_j)\).

• That is, \(v_j\) and all of its neighbors are of degree \(\Delta(G)\). Repeating the same argument for a neighbor \(v_i\) implies that \(v_i\)’s neighbors are also of degree \(\Delta(G)\). We continue to repeat the argument to obtain that the graph is regular.

**Multiplicity**

• The previous proof also shows that any eigenvector \((x_1, \ldots, x_n)\) of the eigenvalue \(d\) satisfies \(x_1 = x_2 = \cdots = x_n\).

• Thus, the space of eigenvectors of the eigenvalue \(d\) is of dimension 1 (that is, \(d\) has multiplicity 1).
The Spectrum of the Petersen Graph

- The Petersen graph $G = (V, E)$ is a 3-regular graph with 10 vertices.
  - We know that it has eigenvalue 3 with eigenvector $1_{10}$.
- To find the other eigenvalues, we notice some useful properties:
  - If $(u, v) \in E$ then $u$ and $v$ have no common neighbors.
  - If $(u, v) \notin E$ then $u$ and $v$ have exactly one common neighbor.

The Adjacency Matrix

- The number of neighbors shared by $v_i$ and $v_j$ is $(A^2)_{ij}$. That is
  $$(A^2)_{ij} = \begin{cases} 
  3, & \text{if } i = j, \\
  0, & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\
  1, & \text{if } i \neq j \text{ and } (v_i, v_j) \notin E. 
  \end{cases}$$
- That is, $A^2 + A - 2I = 1_{n \times n}.$
The Additional Eigenvalues

- We have $A^2 + A - 2I = 1_{n \times n}$.
- Since 3 is an eigenvalue with eigenvector $1_n$, the other eigenvectors are orthogonal to $1_n$.
- Thus, for an eigenvector $v$ of eigenvalue $\lambda \neq 3$: 
  $$1_{n \times n}v = 0_n.$$ 
- That is, $(A^2 + A - 2I)v = 0_n$.
- If $v$ is an eigenvector of $\lambda$ then we have 
  $$\lambda^2 v + \lambda v - 2v = 0_n.$$ 
- Thus, the additional eigenvalues of $A$ are 1, -2.

$$spec(G) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & m_2 & m_3 \end{pmatrix}.$$ 

The Multiplicities

- $spec(G) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & m_2 & m_3 \end{pmatrix}$.
- Recall that $\sum_{i=1}^{10} \lambda_i = trace(A) = 0$.
- That is, $3 + m_2 - 2m_3 = 0$.
- Combining this with $m_2 + m_3 = 9$ and $m_2, m_3 \geq 0$, we obtain the unique solution $m_2 = 5, m_3 = 4$.
- $spec(G) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$. 
Petersen Graph and $K_{10}$

**Problem.** Can we partition the edges of $K_{10}$ into three disjoint sets, such that each set forms a Petersen graph?

- $K_{10}$ has $\binom{10}{2} = 45$ edges, and the Petersen graph has 15.
- In $K_{10}$ every vertex is of degree 9, and in the Petersen graph 3.

**Disproof**

Assume, **for contradiction**, that the partition exists, and let $A, B, C$ be the adjacency matrices of the three copies of the Petersen graph.

- The adjacency matrix of $K_{10}$ is $1_{10 \times 10} - I$.
- That is, $A + B + C = 1_{10 \times 10} - I$.
- $V_A, V_B$ – the vector subspaces of eigenvectors corresponding to the eigenvalue 1 in $A$ and $B$.
- We know that $\dim V_A = \dim V_B = 5$.
- Since both $V_A$ and $V_B$ are orthogonal to $1_{10}$, they are not disjoint (otherwise we would have a set of 11 orthogonal vectors in $\mathbb{R}^{10}$).
Disproof (cont.)

• $A, B, C$ – the adjacency matrices of the three copies of the Petersen graph in $K_{10}$.

• $A + B + C = 1_{n \times n} - I$.

• $V_A, V_B$ – the vector subspaces of eigenvectors corresponding to the eigenvalue 1 in $A$ and $B$.

• $V_A$ and $V_B$ are not disjoint.

• Let $z \in V_A \cap V_B$. Since every vector in $V_A$ and $V_B$ is orthogonal to $1_n$, so is $z$.

• We have
  $$Cz = (1_{n \times n} - I - A - B)z = 0 - z - Az - Bz = -3z.$$  

• **Contradiction since -3 is not an eigenvalue of $C$.**

Four Things You Did Not Know About The Petersen Graph

• It has 15 edges and **2000 spanning trees**.

• It is the smallest 3-regular graph of girth 5 (this is called a **(3,5)-cage graph**).

• It likes gardening, ballet, and building airplane models.

• It has gotten divorced three times.
Moore Graphs

- A Moore graph is a graph that is $d$-regular, of diameter $k$, and whose number of vertices is

$$1 + d \sum_{i=0}^{k-1} (d - 1)^i.$$ 

- As can be easily checked, this is the minimum possible number of vertices of any graph of diameter $k$ and minimum degree $d$.

Examples of Moore Graphs

- A Moore graph is a graph that is $d$-regular, of diameter $k$, and whose number of vertices is

$$1 + d \sum_{i=0}^{k-1} (d - 1)^i.$$ 

- What Moore graphs do we know?
  - The Petersen graph is 3-regular, of diameter 2, and contains $1 + 3 \sum_{i=0}^{1} (3 - 1)^i = 10$ vertices.
Examples of Moore Graphs

- A **Moore graph** is a graph that is \(d\)-**regular**, of diameter \(k\), and whose number of vertices is
  \[
  1 + d\sum_{i=0}^{k-1} (d - 1)^i.
  \]
- The **Petersen graph** is 3-regular, of diameter 2, and contains \(1 + 3 \sum_{i=0}^{1} (3 - 1)^i = 10\) vertices.
- Is there a Moore graph that is 2-regular and of diameter 2? \(C_5\)

Moore Graphs of Diameter 2 and Girth 5

- Recall that the **girth** of a graph is the length of the shortest cycle in it.
- **Theorem.** There exist \(d\)-regular Moore graphs with diameter 2 and girth 5 only for \(d = 2,3,7,\) and possibly 57.
  - The case of \(d = 57\) is an open problem.
  - If it exists, it has 3250 vertices, and 92,625 edges.

The case of \(d = 7\)
- \( G = (V, E) \) – a \( d \)-regular graph of diameter 2, girth 5, and with

\[
|V| = 1 + d \sum_{i=0}^{1} (d - 1)^i = 1 + d^2.
\]

- **Since the girth is five**, if \( (v_i, v_j) \in E \) then \( v_i \) and \( v_j \) have no common neighbors.

- **Since the diameter is two**, if \( (v_i, v_j) \notin E \) then \( v_i \) and \( v_j \) have exactly one common neighbor.

- Thus, the adjacency matrix \( A \) satisfies

\[
(A^2)_{ij} = \begin{cases} 
  d, & \text{if } i = j, \\
  0, & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\
  1, & \text{if } i \neq j \text{ and } (v_i, v_j) \notin E.
\end{cases}
\]

That is, we have \( A^2 + A - (d - 1)I = 1_{n \times n} \).

**Proof (cont.)**

- \( G = (V, E) \) – a \( d \)-regular graph of diameter 2, girth 5, and with \( |V| = 1 + d^2 \).

- \( A^2 + A - (d - 1)I = 1_{n \times n} \).

- \( \lambda \neq d \) – an eigenvalue of \( A \) with eigenvector \( v \).

  Since \( \lambda \neq d \), \( v \) is orthogonal to \( 1_n \). Thus

\[
(A^2 + A - (d - 1)I)v = 0_n, \quad \text{or} \quad \lambda^2 v + \lambda v - (d - 1)v = 0.
\]

- This implies that \( \lambda^2 + \lambda - d + 1 = 0 \), so

\[
\lambda_{2,3} = \frac{-1 \pm \sqrt{1 + 4(d - 1)}}{2} = \frac{-1 \pm \sqrt{4d - 3}}{2}.
\]

- We thus have \( \text{spec}(G) = (d, \lambda_2, \lambda_3) \).
Finding the Multiplicities

- We have \( \lambda_2 = -\frac{1 + \sqrt{4d - 3}}{2}, \lambda_3 = -\frac{1 - \sqrt{4d - 3}}{2}, \)
  
  \[ \text{spec}(G) = \begin{pmatrix} d & \lambda_2 & \lambda_3 \\ 1 & m_2 & m_3 \end{pmatrix}. \]

\[ 0 = \text{trace}(A) = d + \lambda_2 m_2 + \lambda_3 m_3 \]
\[ = d - \frac{m_2 + m_3}{2} + \frac{m_3 - m_2}{2} \sqrt{4d - 3}. \]

- Since \( m_2 + m_3 = n - 1 = d^2 \), we have
  
  \[ d^2 - 2d = (m_3 - m_2)\sqrt{4d - 3}. \]

- This can happen if either \( m_2 = m_3 \) or \( 4d - 3 = s^2 \) for some integer \( s \).
- If \( m_2 = m_3 \) then \( d^2 - 2d = 0 \), implying \( d = 2 \).

The Case of \( 4d - 3 = s^2 \)

\[ d^2 - 2d = (m_3 - m_2)\sqrt{4d - 3}. \]

- Assume that \( 4d - 3 = s^2 \) for some integer \( s \).
  That is, \( d = (s^2 + 3)/4 \).
- Substituting into the above equation:
  
  \[ \frac{(s^2 + 3)^2}{16} - \frac{2s^2 + 6}{4} = (m_3 - m_2)s. \]

- Setting \( m_3 - m_2 = 2m_3 - d^2 \), we have
  
  \[ s^4 + 6s^2 + 9 - 8s^2 - 24 = 32m_3 s - s^5 - 6s^3 - 9s. \]
  \[ s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32m_3)s = 15. \]

- So \( s \) must divide 15, and we get \( s \in \{1,3,5,15\} \),
  which implies \( d \in \{1,3,7,57\} \).
- The case \( d = 1 \) leads to \( K_2 \). Not a Moore graph.
Recall: Independent Sets

- Consider a graph $G = (V, E)$. An independent set in $G$ is a subset $V' \subset V$ such that there is no edge between any two vertices of $V'$.
- Finding a maximum independent set in a graph is a major problem in theoretical computer science.
  - No polynomial-time algorithm is known.

Past Bounds

- Let $G = (V, E)$ be a graph.
- Already proved:
  - $G$ has an independent set of size at least
    $$\sum_{v \in V} \frac{1}{1 + \deg v}.$$  
  - If $|E| = |V| \cdot \frac{d}{2}$, then $G$ has an independent set of size at least $|V|/2d$. 
An Upper Bound

- **Theorem.** For a $d$-regular graph $G = (V, E)$ with smallest (most negative) eigenvalue $\lambda_n$, the size of the largest independent set of $G$ is at most $\frac{n}{1 - d/\lambda_n}$.

- **Example.**
  - $\text{spec}(C_4) = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$.
  - So at most $\frac{4}{1 - \frac{2}{-2}} = 2$.

Recall: The Rayleigh Quotient

- The **Rayleigh quotient** is $R(A, x) = R(x)$
  $$R(x) =\frac{x^T Ax}{x^T x}$$ for $n \times n$ matrix $A$ and $x \in \mathbb{R}^n$.

- **Lemma.** Let $A$ be a real symmetric $n \times n$ matrix. Then $R(x)$ attains its maximum and minimum at eigenvectors of $A$. (We do not prove the lemma.)

- **Question.** What is $R(x)$ when $x$ is an eigenvector of eigenvalue $\lambda$?
  - $\frac{x^T (Ax)}{x^T x} = \frac{x^T \lambda x}{x^T x} = \lambda$.
  - Thus, the min and max values of $R(x)$ are the min and max eigenvalues of $A$. 
\( \lambda_n \) – the most negative eigenvalue of \( G \).

\( S \) – a largest independent set of \( G \).

\( 1_S = (x_1, ..., x_n) \) – a vector with \( x_i = 1 \) if \( v_i \in S \) (otherwise \( x_i = 0 \)).

\( y = n1_S - 1_n \cdot |S| \).

\[ y^T Ay = n^2 \cdot 1_S^T A 1_S - 2|S|n \cdot 1_S^T A 1_n + |S|^2 \cdot 1_n^T A 1_n. \]

Since \( S \) is an independent set, we have

\[ 1_S^T A 1_S = \sum_{i,j \in S} A_{ij} = 0. \]

Since \( G \) is \( d \)-regular, \( 1_S^T A 1_n = 1_S^T \cdot d1_n = d|S| \), and also \( 1_n^T A 1_n = 1_n^T \cdot d1_n = dn. \)

Combining the above, we have

\[ y^T Ay = 0 - 2|S|n \cdot d|S| + |S|^2 \cdot dn = -|S|^2dn. \]

\[ y^T y = n^2 1_S^T 1_S - 2|S|n1_S^T 1_n + |S|^2 1_n^T 1_n \]

\[ = n^2|S| - 2|S|^2n + |S|^2n = |S|n(n - |S|). \]

**Completing the Proof**

- By the lemma, we have
  \[ \frac{y^T Ay}{y^T y} \geq \lambda_n. \]

We have

\[ y^T Ay = -|S|^2dn. \]

\[ y^T y = |S|n(n - |S|). \]

Thus

\[ \lambda_n \leq \frac{-|S|^2dn}{|S|n(n - |S|)} = \frac{-d|S|}{n - |S|}. \]

\[ \lambda_n(n - |S|) \leq -d|S| \quad \rightarrow \quad |S| \leq \frac{n}{1 - d/\lambda_n} \]
The End