Crossing Numbers

- Given a graph $G = (V, E)$, the **crossing number** of $G$, denoted $\text{cr}(G)$ is the minimum number of crossings that a planar drawing of $G$ can have.
- What are the graphs that have a crossing number of zero? **The planar graphs**.
- What is the crossing number of $K_5$? $1$. 
The Crossing Number of $K_6$

- **Problem.** Find the crossing number of $K_6$.
- **Solution.**
  - Recall that any planar graph with $n$ vertices has at most $3n - 6$ edges.
  - That is, any planar subgraph of $K_6$ has at most 12 edges.

![Diagram](image)

**Solution (cont.)**

- We can draw at most 12 edges of $K_6$ without a crossing.
  - $K_6$ has 15 edges, and adding each of the three remaining edges yields at least one crossing.
  - So $cr(K_6) \geq 3$.
  - The following figure shows that $cr(K_6) = 3$. 

![Diagram](image)
A First Bound

**Claim.** For any graph $G = (V, E)$, we have $cr(G) \geq |E| - (3|V| - 6)$.

**Proof.**
- Consider a drawing of $G$ that minimizes the number of crossings.
- We first draw a maximum plane subgraph that is contained in this drawing. This subgraph has at most $3|V| - 6$ edges.
- Adding every additional edge increases the number of crossings by at least one. There are at least $|E| - (3|V| - 6)$ such edges.

Is This a Good Bound?

**For simple graphs, the bound**

$$cr(G) \geq |E| - 3|V| + 6$$

**is never larger than $|V|^2/2$.**
- Is this close to the maximum number of crossings that is possible?
- To find the maximum possible number of crossings, we consider $cr(K_n)$. The above bound implies

$$cr(K_n) \geq \frac{n(n - 1)}{2} - 3n + 6 \approx \frac{n^2}{2}.$$
**Estimating $cr(K_n)$**

- **Theorem.** For sufficiently large $n$, we have
  \[
  \frac{n^4}{120} - cn^3 \leq cr(K_n) \leq \frac{n^4}{24} + cn^3,
  \]
  for some constant $c$.

- **Upper bound**
  - **Trivial!**
    - Every crossing is the intersection of two edges, which are defined by four vertices.
    - The number of ways to choose four vertices is
      \[
      \binom{n}{4} \approx \frac{n^4}{24}.
      \]
Lower Bound

- Consider a drawing of $K_n$ that minimizes the number of crossings.
  - Removing any one vertex results in a drawing of $K_{n-1}$. This drawing has at least $cr(K_{n-1})$ crossings.
  - We have $n$ different drawings of $K_{n-1}$, and together they contain at least $n \cdot cr(K_{n-1})$ crossings.
  - Each crossing is counted exactly $n - 4$ times. Thus, we have
    \[
    (n - 4) \cdot cr(K_n) \geq n \cdot cr(K_{n-1}).
    \]

Lower Bound (cont.)

- We prove by **induction on** $n$ that $cr(K_n) \geq \frac{1}{5} \binom{n}{4}$.
  - **Induction basis.** For $n = 5$, we know that
    \[
    cr(K_5) = 1 = \frac{1}{5} \binom{5}{4}.
    \]
  - **Induction step.** By the previous slide
    \[
    cr(K_n) \geq \frac{n}{n - 4} cr(K_{n-1}) \geq \frac{n}{n - 4} \cdot \frac{1}{5} \binom{n - 1}{4} \geq \frac{n}{n - 4} \cdot \frac{1}{5} \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{4!}
    \]
    \[
    = \frac{n}{n - 4} \cdot \frac{1}{5} \cdot \frac{n - 1}{(n - 3)(n - 2)(n - 4)}
    \]
    \[
    = \frac{1}{5} \binom{n}{4}.
    \]
The Correct Bound

- Somewhat more involved arguments lead to \( cr(K_n) \approx \frac{n^4}{64} \).
- It is conjectured that
  \[
  cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor ,
  \]
  but the problem remains open.
  - This is known as the brick factory problem, since it was asked by Turán while doing forced labor in a brick factory during World War II.

So How Bad is Our Bound?

- We have the bound
  \[ cr(G) \geq |E| - 3|V| + 6. \]
- This implies that \( cr(K_n) \geq \frac{n^2}{2} \).
- We have \( cr(K_n) \approx \frac{n^4}{64} \).
- For large \( n \) the bound is significantly smaller than the actual value.
An Improved Bound

**Theorem.** Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$. Then
\[
\text{cr}(G) \geq \frac{|E|^3}{64|V|^2}.
\]

- We consider a drawing of $G$ with a minimum number of crossings $c$. Set $p = \frac{4|V|}{|E|}$.
- $S \subset V$ – the subset obtained by independently choosing each vertex of $V$ with probability $p$.
- $c_S$ – the number of crossings that remain in the drawing of $G$ after removing $V \setminus S$.
- $G_S = (S, E_S)$ – the subgraph induced on $S$.
- $\mathbb{E}[|S|] = p|V|$.  \( \mathbb{E}[|E_S|] = p^2|E| \).
- $\mathbb{E}[c_S] = p^4c$.

**By linearity of expectation**
\[
\mathbb{E}[c_S - |E_S| + 3|S|] = p^4c - p^2|E| + 3p|V|
\]
\[
= \frac{4^4|V|^4c}{|E|^4} - \frac{16|V|^2}{|E|} + \frac{12|V|^2}{|E|}.
\]
\( G_S = (S, E_S) \) – the subgraph induced on \( S \).

\( c_S \) – the number of crossings that remain in the drawing of \( G \) after removing \( V \setminus S \).

\[
E[c_S - |E_S| + 3|S|] = \frac{4^4|V|^4c}{|E|^4} - \frac{4|V|^2}{|E|}.
\]

Thus, there exists a set \( S \) with

\[
c_S - |E_S| + 3|S| \leq \frac{4^4|V|^4c}{|E|^4} - \frac{4|V|^2}{|E|}.
\]

By the weak bound, \( c_S \geq |E_S| - 3|S| + 6 \).

Thus, \( \frac{4^4|V|^4c}{|E|^4} \geq \frac{4|V|^2}{|E|} + 6 > \frac{4|V|^2}{|E|} \). That is,

\[
c > \frac{|E|^3}{64|V|^2}.
\]

A Minor Detail

Where in the proof did we use the restriction \( |E| \geq 4|V| \)?

- The probability for choosing a vertex is \( p = \frac{4|V|}{|E|} \). If \( |E| < 4|V| \), this is not well defined.
Is This Bound Better?

- We have \( cr(G) \geq \frac{|E|^3}{64|V|^2} \).
  - That is \( cr(K_n) \geq \frac{\left(\frac{n^2}{2}\right)^3}{64n^2} = \frac{n^4}{2^9} \).
  - Recall that \( cr(K_n) \approx \frac{n^4}{64} \).
  - Even though there is a gap in the constants, the dependency on \( n \) is correct.

Point-Line Incidences

- \( L \) – a set of \( n \) lines.
- \( P \) – a set of \( m \) points.
- An incidence: \( (p, \ell) \in P \times L \) so that \( p \in \ell \).
Lower Bound

- **Erdős.** By taking a $\sqrt{m} \times \sqrt{m}$ integer lattice and the $n$ lines that contain the largest number of points, we have $c(m^{2/3}n^{2/3} + m + n)$ incidences.

The Szemerédi–Trotter Theorem

- **Theorem.** The number of incidences between any set $P$ of $m$ points and any set $L$ of $n$ lines is at most $c(m^{2/3}n^{2/3} + m + n)$. 
• $I$ – the number of incidences.

• We build a graph.
  ◦ A vertex for every point.
  ◦ An edge between two vertices if they are consecutive on a line.

• A line that is incident to $k$ points yields $k - 1$ edges. Thus, the number of edges is $I - n$.
  ◦ By the crossing lemma, the number of crossings in the graph is at least \( \frac{(I-n)^3}{64m^2} \).
  ◦ Since every two lines intersect at most once, the number of crossings is less than \( \frac{n^2}{2} \).

\[
\frac{(I-n)^3}{64m^2} < \frac{n^2}{2} \quad \rightarrow \quad I < 32^{\frac{3}{2}m^{\frac{3}{2}n^{\frac{3}{2}}}} + n
\]

A Minor Issue

• The lower bound on the number of crossings applies only when \( |E| \geq 4|V| \).
  ◦ Since \( |E| = I - n \), if \( |E| < 4|V| \) then
    \( I < n + 4m \).
The Unit Distances Problem

- **Problem (Erdős `46).** How many pairs of points in a set of \( n \) points could be at unit distance from each other?
  - By taking \( n \) points evenly spaced on a line, we have \( n - 1 \) unit distances.

Early Results

- **Erdős** showed that a \( \sqrt{n} \times \sqrt{n} \) square lattice with the right distances determines \( n^{1+c/\log \log n} \) unit distances, for some constant \( c \).
- Erdős also proved that any set of \( n \) points determines at most \( cn^{3/2} \) unit distances.
An Improved Result

- Although in the past 70 years MANY top combinatorists worked on the problem, only one work managed to improve the bound (Spencer, Szemerédi, and Trotter 1984).

- **Theorem.** Every set of \( n \) points determines at most \( cn^{4/3} \) unit distances.

Incidences with Circles

- Given a set of circles and a set of points, an incidence is a pair \((p, C)\) where \( p \) is a point, \( C \) is a circle, and \( p \) is contained in \( C \).

- **11** incidences are in the figure.
Unit Distances and Unit Circles

- We place a unit circle around every point.
- The number of point-circle incidences is twice the number of unit distances.
- Thus, it suffices to find an upper bound for the number of incidences between $n$ points and any $n$ unit circles.

Incidence Bound

- **Theorem.** There are at most $cn^{4/3}$ incidences between any set $P$ of $n$ points and any set $C$ of $n$ unit circles.
Building a Graph

- We build a graph.
  - A vertex for every point.
  - An edge between two points if they are consecutive along at least one circle.
  - A circle that is incident to $k$ points yields at least $k - 1$ edges.
  - An edge can originate from at most two circles.

Double Counting Crossings

- $I$ – the number of point-circle incidences.
- We have a graph with $n$ vertices and at least $(I - n)/2$ edges.
- The number of crossings in the graph at least
  \[
  \frac{|E|^3}{64|V|^2} \geq \frac{(I - n)^3}{2^9 n^2}.
  \]
- Since any two circles intersect at most twice, the number of crossings is at most $n^2$.
- Combining the two implies $n^2 \geq \frac{(I - n)^3}{2^9 n^2}$. That is $(I - n)^3 \leq 2^9 n^4$ or $I \leq 2^3 n^3 + n$. 
The End

Crossings will appear in the next Piled Higher and Deeper movie!

“Girl, that — is so non-planar you can call it $K_5$”