Basic Probability

- A discrete probability space is a finite set $\Omega$. Each $\omega \in \Omega$ is called an elementary event, and has a certain probability $\Pr[\omega] \in [0, 1]$, such that $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.
- Any subset $A \subseteq \Omega$ is an event, of probability $\Pr[A] = \sum_{\omega \in A} \Pr[\omega]$.
- A union of events corresponds to OR and an intersection of events corresponds to AND.
Independent Events

- Two events $A, B \subseteq \Omega$ are independent if $\Pr[A \cap B] = \Pr[a] \cdot \Pr[b]$.

- Example. We flip two fair coins.
  - Let $\omega_{i,j}$ be the elementary event that coin $A$ landed on $i$ and coin $B$ on $j$, where $i, j \in \{h, t\}$. Each of the four events has a probability of 0.25.
  - The event where coin $A$ lands on heads is $a = \{\omega_{h,t}, \omega_{h,h}\}$. For $B$ it is $b = \{\omega_{t,h}, \omega_{h,h}\}$.
  - The events are independent since $\Pr[a \text{ and } b] = \Pr[\omega_{h,h}] = 0.25 = \Pr[a] \cdot \Pr[b]$.

(Discrete) Uniform Distribution

- In a uniform distribution we have a set $\Omega$ of elementary events, each occurring with probability $\frac{1}{|\Omega|}$.
  - For example, when flipping a fair die, we have a uniform distribution over the six possible results.
Union Bound

- For any two events $A, B$, we have
  \[ \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]. \]

- This immediately implies that
  \[ \Pr[A \cup B] \leq \Pr[A] + \Pr[B], \]
  where equality holds iff $A, B$ are disjoint.

- **Union bound.** For any finite set of events $A_1, \ldots, A_k$, we have
  \[ \Pr[\bigcup_i A_i] \leq \sum_i \Pr[A_i]. \]

Recall: Ramsey Numbers

- $R(p, p)$ is the smallest number $n$ such that each blue-red edge coloring of $K_n$ contains a monochromatic $K_p$.

- **Theorem.** $R(p, p) > 2^{p/2}$.
  - In the previous class we provided one proof for this.
  - Now we provide another proof, using probability.
Probabilistic Proof

- For some \( n \), we color the edges of \( K_n \).
  - Each edge is independently and uniformly colored either blue or red.
  - For any fixed set \( S_\alpha \) of \( p \) vertices, the probability that it forms a monochromatic \( K_p \) is \( 2^{1-\binom{p}{2}} \).
  - There are \( \binom{n}{p} \) possible sets of \( p \) vertices. By the union bound, the probability that there is a monochromatic \( K_p \) is at most
    \[
    \sum_{\alpha} 2^{1-\binom{p}{2}} = \binom{n}{p} 2^{1-\binom{p}{2}}.
    \]

Proof (cont.)

- For some \( n \), we color the edges of \( K_n \).
  - Each edge is colored blue with probability of 0.5, and otherwise red.
  - The probability for a monochromatic \( K_p \) is
    \[
    \leq \sum_{\alpha} 2^{1-\binom{p}{2}} = \binom{n}{p} 2^{1-\binom{p}{2}}.
    \]
  - If \( n \leq 2^{p/2} \), this probability is smaller than 1.
  - In this case, the probability that we do not have any monochromatic \( K_p \) is positive, so there exists a coloring of \( K_n \) with no such \( K_p \).
Non-Constructive Proofs

- We proved that there exists a coloring of $K_n$ with no monochromatic $K_p$, but we have no idea how to find this coloring.
- Such a proof is called non-constructive.
- The probabilistic method often proves the existence of objects with surprising properties, but we still have no idea how they look like.

A Tournament

- We have $n$ people competing in thumb wrestling.
  - Every pair of contestants compete once.
  - How can we decide who the overall winner is?
- We build a directed graph:
  - A vertex for every team.
  - An edge between every two vertices, directed from the winner to the loser.
  - An orientation of $K_n$ is called a tournament.
The King of the Tournament

• The winner can be the vertex with the maximum outdegree (the contestant winning the largest number of matches), but it might not be unique.
• A king is a contestant $x$ such that for every other contestant $y$ either $x \rightarrow y$ or there exists $z$ such that $x \rightarrow z \rightarrow y$.
• Theorem. Every tournament has a king.

Proof

• $D^+(v)$ – the number of vertices reachable from $v$ by a path of length $\leq 2$.
• Let $v$ be a vertex that maximizes $D^+(v)$.
  ◦ Assume, for contradiction, that $v$ is not a king.
  ◦ Then there exists $u$ such that $u \rightarrow v$ and there is no path of length two from $v$ to $u$.
  ◦ That is, for every $w$ such that $v \rightarrow w$, we also have $u \rightarrow w$.
  ◦ But this implies that $D^+(u) \geq D^+(v) + 1$, contradicting the maximality of $v$!
The $S_k$ Property

- We say that a tournament $T$ has the $S_k$ property if for every subset $S$ of $k$ participants, there exists a participant that won against everyone in $S$.
  - Formally, this is an orientation of $K_n$, such that for every subset $S$ of $k$ vertices there exists a vertex $v$ with an edge from $v$ to every vertex of $S$.

- Example. A tournament with the $S_1$ property.

Tournaments with the $S_k$ Property

- Theorem. If $\binom{n}{k} \left(1 - 2^{-k}\right)^{n-k} < 1$ then there is a tournament on $n$ vertices with the $S_k$ property.

- Proof.
  - For some $n$ satisfying the above, we randomly orient $K_n = (V, E)$, such that the orientation of every $e \in E$ is chosen uniformly.
  - Consider a subset $S \subset V$ of $k$ vertices. The probability that a given vertex $v \in V \setminus S$ does not beat all of $S$ is $1 - 2^{-k}$.
Proof (cont.)

- Consider a subset $S \subset V$ of $k$ vertices. The probability that a specific vertex $v \in V \setminus S$ does not beat every vertex of $S$ is $1 - 2^{-k}$.

- $A_S$ – the event of $S$ not being beat by any vertex of $V \setminus S$.

- We have $\Pr[A_S] = (1 - 2^{-k})^{n-k}$, since we ask for $n - k$ independent events to hold.

- By the union bound, we have

$$\Pr \left[ \bigvee_{S \subset V \atop |S|=k} A_S \right] \leq \sum_{S \subset V \atop |S|=k} \Pr[A_S] = \binom{n}{k} (1 - 2^{-k})^{n-k} < 1.$$ 

Completing the Proof

- $A_S$ – the event of $S$ not being beat by any vertex of $V \setminus S$.

- we have

$$\Pr \left[ \bigvee_{S \subset V \atop |S|=k} A_S \right] < 1.$$ 

- That is, there is a positive probability that every subset $S \subset V$ of size $k$ is beat by some vertex of $V \setminus S$. So such a tournament exists.
Which NBA Player is Related to Mathematics?

Michael Jordan  Shaquille O'Neal  LeBron James

Intersecting Sets

• Given two subsets $A, B \subset \{0,1,2, \ldots, n-1\}$, we say that $A$ and $B$ intersect if $A \cap B \neq \emptyset$.

• **Question.** Consider a set $S$ of subsets of $k$ elements of $\{0,1,2, \ldots, n-1\}$, such that every two subsets of $S$ intersect.

• How large can $|S|$ be?
  - If $k > \frac{n}{2}$ then $S$ can contain all $\binom{n}{k}$ subsets of $k$ elements.
  - Thus, we assume that $k \leq n/2$. 


Large Intersecting Families

- We can take all of the $k$-element subsets of $\{0,1,2,\ldots,n-1\}$ that contain 1.
  - Each pair of such subsets intersect.
  - The number of such subsets is $\binom{n-1}{k-1}$.
- In the special case of $n = 2k$, we can take all of the subset that do not contain 1.
  - Example. For $k = 2$ and $n = 4$, we take \{2,3\}, \{0,3\}, \{0,2\}.
  - Each two such subsets intersect.
  - The number of such subsets is $\binom{n-1}{k-1}$.

Erdős–Ko–Rado Theorem

- For any $k \leq \frac{n}{2}$, we know that there exists a set of $\binom{n-1}{k-1}$ intersecting subsets of size $k$.
  - Can we obtain a larger intersecting family of such subsets?
- Theorem. For $n \geq 2k$, every family $F$ of intersecting $k$-element subsets of $\{0,1,2,\ldots,n-1\}$, we have $|F| \leq \binom{n-1}{k-1}$.
A Quick Lemma

- **Lemma.** For $0 \leq s \leq n - 1$ set $A_s = \{s, s + 1, \ldots, s + k - 1\}$ with addition $mod\ n$. Then an intersecting family $F$ can contain at most $k$ of the $A_s$’s.

- **Proof.** Fix some $A_s \in F$.
  - The subsets that intersect it are $A_{s-k+1}, A_{s-k+2}, \ldots, A_{s+k-2}, A_{s+k-1}$.
  - We arrange these subsets into $k - 1$ pairs $\{A_{s-k+i}, A_{s+i}\}$.
  - The two subsets in each pair are disjoint, so $F$ contains at most one of them.

Proving the Theorem

- Consider an intersecting family $F$.
  - We **uniformly and independently** choose a permutation $\sigma$ of $\{0, 1, \ldots, n - 1\}$ and a number $i \in \{0, 1, \ldots, n - 1\}$.
  - Let $A_i = \{\sigma(i), \sigma(i + 1), \ldots, \sigma(i + k - 1)\}$ (as before, under addition $mod\ n$).
  - By applying the lemma from the previous slide with respect to the new ordering defined by the permutation, we have

$$Pr[A_i \in F] \leq \frac{k}{n}.$$
Consider an intersecting family $F$.

- We uniformly and independently choose a permutation $\sigma$ of $\{0,1, ..., n-1\}$ and a number $i \in \{0,1, ..., n-1\}$.
- For $A_i = \{\sigma(i), \sigma(i+1), ..., \sigma(i+k-1)\}$, we have $\Pr[A_i \in F] \leq \frac{k}{n}$.
- On the other hand, since $A_i$ is chosen uniformly out of the $\binom{n}{k}$ possible subsets, we have $\Pr[A_i \in F] = \frac{|F|}{\binom{n}{k}}$.

Combining the two bounds implies

$$|F| \leq \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}.$$

**Random Variables**

- A random variable is a function from the set of possible events to $\mathbb{R}$.

**Example.** Say that we flip five coins.

- We can define the random variable $X$ to be the number of coins that landed on heads.
- We can define the random variable $Y$ to be the percentage of heads in the tosses.
- Notice that $Y = 20X$. 
Indicator Random Variables

- An *indicator random variable* is a random variable $X$ that is either 0 or 1, according to whether some event happens or not.

**Example.** We toss a fair die.

- We can define the six indicator variable $X_1, \ldots, X_6$ such that $X_i = 1$ iff the result of the roll is $i$.

Expectation

- The *expectation* of a random variable $X$ is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \Pr[\omega].$$

- Intuitively, $E[X]$ is the expected value of $X$ in the long-run average value of repetitions of the experiment it represents.
Expectation Example

- We roll a fair six-sided die.
  - Let $X$ be a random variable that represents the outcome of the roll.
  
  $$E[X] = \sum_{\omega \in \Omega} X(\omega) \Pr[\omega] = \sum_{i \in \{1,\ldots,6\}} i \cdot \frac{1}{6} = 3.5$$

While a prisoner of war during World War II, J. Kerrich conducted an experiment in which he flipped a coin 10,000 times and kept a record of the outcomes. A portion of the results is given in the table below.

<table>
<thead>
<tr>
<th>Number of Tosses</th>
<th>Number of Heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>25</td>
</tr>
<tr>
<td>100</td>
<td>44</td>
</tr>
<tr>
<td>500</td>
<td>255</td>
</tr>
<tr>
<td>1,000</td>
<td>502</td>
</tr>
<tr>
<td>5,000</td>
<td>2,533</td>
</tr>
<tr>
<td>10,000</td>
<td>5,067</td>
</tr>
</tbody>
</table>
Linearity of Expectation

- If $X$ is a random variable, then $5X$ is a random variable with a value five times that of $X$.
- **Lemma.** Let $X_1, X_2, ..., X_k$ be a collection set of random variables over the same discrete probability. Let $c_1, ..., c_k$ be constants. Then

$$E[c_1X_1 + c_2X_2 + \cdots + c_kX_k] = \sum_{i=1}^{k} c_i E[X_i].$$

Fixed Elements in Permutations

- Let $\sigma$ be a uniformly chosen permutation of $\{1, 2, ..., n\}$.
  - For $1 \leq i \leq n$, let $X_i$ be an **indicator variable** that is 1 if $i$ is fixed by $\sigma$.
  - $E[X_i] = \Pr[\sigma(i) = i] = \frac{(n-1)!}{n!} = \frac{1}{n}$.
  - Let $X$ be the number of fixed elements in $\sigma$.
  - We have $X = X_1 + \cdots + X_n$.
  - **By linearity of expectation**
    $$E[X] = \sum_{i} E[X_i] = n \cdot \frac{1}{n} = 1.$$
Hamiltonian Paths

- Given a directed graph $G = (V, E)$, a Hamiltonian path is a path that visits every vertex of $V$ exactly once.
  - Major problem in theoretical computer science: Does there exist a polynomial-time algorithm for finding whether a Hamiltonian path exists in a given graph.

Hamiltonian Paths in Tournaments

- Theorem. There exists a tournament $T$ with $n$ players that contains at least $n! 2^{-n+1}$ Hamiltonian paths.
Proof

- We **uniformly** choose an orientation of the edges of $K_n$ to obtain a tournament $T$.
  - There is a **bijection** between the possible Hamiltonian paths and the permutations of $\{1, 2, \ldots, n\}$. Every possible path defines a unique permutation, according to the order in which it visits the vertices.
  - For a permutation $\sigma$, let $X_\sigma$ be an **indicator variable** that is 1 if the path corresponding to $\sigma$ exists in $T$.
  - We have $E[X_\sigma] = \Pr[X_\sigma = 1] = 2^{-n+1}$.

- We **uniformly** choose an orientation of the edges of $K_n$ to obtain a tournament $T$.
  - For a permutation $\sigma$, let $X_\sigma$ be an **indicator variable** that is 1 if the path corresponding to $\sigma$ exists $T$.
  - We have $E[X_\sigma] = \Pr[X_\sigma = 1] = 2^{-n+1}$.
  - Let $X$ be a random variable of the number of Hamiltonian paths in $T$. Then $X = \sum_\sigma X_\sigma$.
    
    $E[X] = \sum_\sigma E[X_\sigma] = n! \cdot 2^{-n+1}$.
  - Since this is the expected number of paths in a uniformly chosen tournament, there must is a orientation with at least as many paths.
The End: Michael Jordan

- **Math major** in college.
  - In his junior year he switched to cultural geography (whatever that means...).