Ma/CS 6b
Class 15: Ramsey Theory 2

Recall: Ramsey Numbers

- \( r, p_1, ..., p_k \) – positive integers.
- The **Ramsey number** \( R(p_1, ..., p_k; r) \) is the minimum integer \( N \) such that every coloring of \( \binom{[N]}{r} \) using \( k \) colors yields an \( i \)-homogeneous set of size \( p_i \), for some \( 1 \leq i \leq k \).
Example

• When $r = 2$, we are coloring the edges of a graph. The expression $N = R(3, 3, 3; 2)$ means that every coloring of the edges of $K_N$ using four colors contains a monochromatic triangle.

Example 2

• When $r = 3$, we color triples of vertices.
  ◦ Can be thought of as coloring the triangular faces of $K_n$.
  ◦ We are looking for a large subset $S$ of the vertices, such that each triangle that is spanned by three vertices of $S$ has the same color.
Recall: Ramsey’s Theorem

- **Theorem.** For any positive integers $r, p_1, \ldots, p_k$, the Ramsey number $R(p_1, \ldots, p_k; r)$ is finite.
  - One way to think of the theorem: in every sufficiently large arbitrary object (i.e., an arbitrary coloring) there must be some order (i.e., a monochromatic subset).

Some History

- In the early 1930’s in Budapest, a few students used to regularly meet on Sundays in a specific city park bench.
- Among the participants were Paul Erdős, George Szekeres, and Esther Klein.
- Klein told the group that for any set of five points with no three on a line, four of the points are the vertices of a convex quadrilateral.
• The convex hull of a point set $S$ is the smallest convex polygon that contains $S$.

• Given a set of five points:
  ◦ If the convex hall of the point set has at least four points in it, then we are done.

  ![Convex Hull Example 1](image1)

  ![Convex Hull Example 2](image2)

  ◦ If the convex hull consists of three points, we consider the line $\ell$ that passes through the two interior points.

  ◦ We take the two interior points and the two points that are on the same side of $\ell$.

The Story Continues

• The Budapest students started to think about whether there exists an $n$ such that every set of $n$ points with no three on a line contains the vertices of a convex pentagon.

  ◦ More generally, does a similar condition hold for every convex $k$-gon?
The Happy Ending Problem

- Even though Erdős was part of the group, the first to prove the claim was Szekeres.
- A couple of years later, Esther Klein, who suggested the problem married Szekeres who solved it.
  - Since then, this problem is called “the happy ending problem”.
  - They lived together up to their 90’s.

The Theorem

- **Theorem.** For every integer $m \geq 3$, there exists an integer $N(m)$ such that any set of $N(m)$ points with no three on a line contains a subset of $m$ points that are the vertices of a convex $m$-gon.
First Claims

• **Straightforward claim.** Any four vertices of a convex \( n \)-gon span a convex quadrilateral.

• **Less straightforward claim.** If every four vertices of an \( n \)-gon \( P \) form a convex quadrilateral, then \( P \) is convex.

Proving the Claim

• **Claim.** If every four vertices of an \( n \)-gon \( P \) form a convex quadrilateral, then \( P \) is convex.

• **Proof.** We assume, for contradiction, that \( P \) is not convex.
  - There is a vertex \( v \) that is not in the convex hull of the vertices of \( P \).
  - Triangulate the convex hull of \( P \).
  - \( v \) together with the three vertices of the triangle containing \( v \) do not span a convex quadrilateral!
Proving the Theorem

- Set \( N(m) = R(m, 5; 4) \).
  - Given a set of \( N(m) \) points with no three on a line, we color every 4-tuple of points.
  - A 4-tuple is colored red if it spans a convex quadrilateral, and otherwise blue.
  - By Ramsey’s theorem, either there is a red subset of size \( m \) and or blue one of size 5.
  - But we proved that for any 5 points there must be a convex (red) quadruple!
  - Thus, there is a red subset of size \( m \), and by the previous claim it spans a convex \( m \)-gon.

The Erdős-Szekeres Conjecture

- In the 1930’s, Erdős and Szekeres proved
  \[ 2^{m-2} + 1 \leq N(m) \leq \binom{2m-4}{m-2} + 1. \]
- In 1997, Tóth and Valtr improved this to
  \[ N(m) \leq \binom{2m-5}{m-2} + 2. \]
- Conjecture (Erdős and Szekeres).
  \[ N(m) = 2^{m-2} + 1. \]
  - Using computers, this was verified for \( m \leq 6 \).
Frank P. Ramsey

- Died in 1930 at the age of 26. By then, he:
  - Wrote mathematical papers, including getting a whole subfield named after him.
  - Wrote several philosophical works. Wittgenstein mentions him in the introduction to his *Philosophical Investigations* as an influence.
  - Wrote several economics papers, as a student to John Maynard Keynes.
  - Had a wife, kids, etc.

Computing Ramsey Numbers

- Similarly to computing $N(m)$, it is extremely difficult to compute exact values of Ramsey numbers.
  - We consider the simplest case, where $r = 2$. That is, we color edges of graphs.
  - Moreover, assume that there are only two colors. We write $R(p, q) = R(p, q; 2)$. 
Fighting Aliens

“Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.”

Joel Spencer

The Known Bounds

- Best known bounds for $R(r, s; 2)$:

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<td>113–298</td>
<td>205–540</td>
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* Taken from Wikipedia
\[ R(3,3) \]
- In the previous class we proved that any red-blue coloring of \( K_6 \) contains a monochromatic triangle.
- This is not the case for \( K_5 \).
- Thus, we have \( R(3,3) = 6 \).

\[ R(2,4) \]
- What is \( R(2,4) \)?
  - Either there is a red \( K_2 \), which is just a red edge, or there is a blue \( K_4 \).
  - This is obviously the case for any coloring of \( K_4 \), but not for any coloring of \( K_3 \).
  - Thus, \( R(2,4) = 4 \).
\[ R(3,4) \]

- The coloring of \( K_8 \) below contains no blue \( K_3 \) and no red \( K_4 \). Therefore, \( R(3,4) > 8 \).

- **Claim.** \( R(3,4) = 9 \).
  - Consider any red-blue coloring of the edges of \( K_9 \), and pick any vertex \( u \).
  - Assume that \( u \) is incident to the vertices \( \{v_1, ..., v_4\} \) by blue edges. Since \( R(2,4) = 4 \), either there is a red \( K_4 \) or there is a blue \( K_3 \) containing \( u \).

\[ R(3,4) \] Analysis

- Consider any red-blue coloring of the edges of \( K_9 \), and pick any vertex \( u \).
  - If \( u \) is incident to four blue edges, we are done.
  - Assume that \( u \) is incident to the vertices \( \{v_1, ..., v_6\} \) by red edges. Since \( R(3,3) = 6 \), either we have a blue \( K_3 \), or we have a red \( K_4 \) containing \( u \).
  - Since \( \deg u = 8 \), it remains to consider the case where \( u \) is incident to three blue edges and to five red edges.
Completing the $R(3,4)$ Analysis

- It remains to consider the case where every vertex of $K_9$ is incident to exactly three red edges.
  - By deleting the blue edges, we remain with nine vertices of degree three.
  - This is impossible since the sum of the degrees must be even! Thus, this case cannot happen.

Asymptotic Bounds

- In the previous class, we already proved $R(p_1, p_2) \leq R(p_1 - 1, p_2) + R(p_1, p_2 - 1)$.

- Using this bound, one can show that $R(p, p) \leq c \frac{4^p}{\sqrt{p}}$ (for some constant $c$).

- Theorem. $R(p, p) > 2^{p/2}$. 
Proof

- Consider all $2^n\binom{n}{2}$ red-blue colorings of $K_n$.
- A subset of $p$ vertices of $K_n$ forms a red or blue $K_p$ in $2^n\binom{n}{2} - \binom{p}{2} + 1$ of these colorings.
- There are at most $\binom{n}{p} 2^n\binom{n}{2} - \binom{p}{2} + 1$ colorings that contain a monochromatic $K_p$.
- Thus, if $\binom{n}{p} 2^n\binom{n}{2} - \binom{p}{2} + 1 < 1$, there must be colorings with no monochromatic $K_p$.
- This happens approximately when $n < 2^{p/2}$, so $R(p, p) > 2^{p/2}$.

Graph Ramsey Theory

- So far we looked for monochromatic copies of some $K_m$ in colorings of $K_n$.
  - Searching monochromatic copies of other types of graphs has also been studied.
  - Given two graphs $G_1, G_2$, we denote by $R(G_1, G_2)$ the minimum number $n$ such that every coloring of $K_n$ contains either a blue copy of $G_1$ or a red copy of $G_2$.
- Example.
  - $P_m$ – a graph that is a path of length $m$.
  - Then $R(P_2, P_2) = 3$. 
The Case of a Tree

**Theorem.** Let $T$ be a tree with $m$ vertices. Then $R(T, K_n) = (m - 1)(n - 1) + 1$.

**Proof.** We begin with a lower bound.

- Consider $n - 1$ copies of $K_{m-1}$ colored completely in blue. Edges between vertices of different copies are colored red.
- This is a set of $(m - 1)(n - 1)$ vertices containing no red $K_n$ and no blue $T$.
- Thus, $R(T, K_n) > (m - 1)(n - 1)$.

Proof: Upper Bound

**Claim.** Let $T$ be a tree with $m$ vertices. Then $R(T, K_n) = (m - 1)(n - 1) + 1$.

**Proof.** We prove the upper bound by induction on $(m, n)$.

- **Induction basis:** if $m = 1$ or $n = 1$, the claim obviously holds.
- **Induction step:** Set $N = (m - 1)(n - 1) + 1$. Consider a coloring of $K_N$ and a vertex $v$.
- If $v$ is incident to more than $(m - 1)(n - 2)$ red edges, by the hypothesis, either the neighbors span a blue $T$ or $v$ and its neighbors span a red $K_n$. 
Proof: Upper Bound (cont.)

- We set $N = (m - 1)(n - 1) + 1$.
- If any vertex is incident to more than $(m - 1)(n - 2)$ red edges, we are done.
- Assume that each vertex is incident to at most $(m - 1)(n - 2)$ red edges.
- That is, every vertex is adjacent to at least $m - 1$ blue edges.
- To complete the proof, we show that any graph with minimum degree at least $m - 1$ contains every tree with $m$ vertices.

- **Claim.** A graph $G$ with minimum degree at least $m - 1$ contains every tree $T$ with $m$ vertices.

- **Proof.** By induction on $m$.
  - **Induction basis.** Obvious for $m = 1$ or 2.
  - **Induction step.** Consider any tree $T$ and any graph $G$ as stated above.
    - Let $v$ be a leaf of $T$, let $u$ be the vertex connected to it, and let $T' = T \setminus \{v\}$.
    - By the hypothesis, $G$ contains a copy $C$ of $T'$. Let $x \in G$ be the vertex corresponding to $u$ in $C$. Since $\deg u \geq m - 1$, it is connected to a vertex $y \notin C$. By considering $y$ as $v$, we obtain a copy of $T$ in $G$. 

![Diagram](attachment:image.png)
Copies of Triangles

• We denote as $mK_3$ a set of $m$ disjoint copies of $K_3$.

• **Theorem.** $R(mK_3, mK_3) = 5m$, for every $m \geq 2$.

• **Proof.** We begin with the lower bound.
  ◦ Consider a red $K_{3m-1}$ and another red $K_{1,2m-1}$. Every other edge in the graph is blue.
  ◦ There are $5m - 1$ vertices, with $m - 1$ disjoint red triangles and $m - 1$ disjoint blue triangles.
  ◦ Thus, $R(mK_3, mK_3) > 5m - 1$.

Illustration

$K_{3m-1}$  $K_{1,2m-1}$
Proof: Upper Bound

- **Theorem.** \( R(mK_3, mK_3) = 5m, \) for \( m \geq 3. \)

- **Proof.** We prove an upper bound by induction on \( m. \)
  
  - **Induction basis.** The case of \( m = 2 \) is not trivial, but we will not do it.
  
  - **Induction step.** Consider a coloring of \( K_{5m}. \)
    - We repeatedly look for a monochromatic triangle and remove its vertices from the graph.
    - Since \( R(3,3) = 6 \), this process continues as long as at least six vertices remain.
    - Since \( 5m - 3m \geq 6 \) for \( m \geq 3 \), we have at least \( m \) disjoint monochromatic triangles.

Proof: Upper Bound (cont.)

- We consider a coloring of \( K_{5m} \) and find \( m \) monochromatic triangles in it.
- If all triangles are of the same color, we are done. Thus, assume that we have a red triangle \( \Delta abc \) and a blue triangle \( \Delta def. \)
- WLOG, assume that at least 5 of the 9 edges between the two triangles are red.
- WLOG, assume that two of these 5 edges meet in \( d. \) We thus have a red triangle and a blue triangle with a common vertex \( d. \)
Proof: Upper Bound (cont.)

- We consider a coloring of $K_{5m}$.
- It remains to consider the case of a red triangle and a blue triangle have a common vertex.
- By the induction hypothesis, the remaining $5m - 5$ vertices contain $m - 1$ disjoint triangles of the same color.
- By adding one of the two triangles, we obtain $m$ triangles of the same color.

The End