The Pigeonhole Principle

- **The pigeonhole principle.** If \( n \) items are put into \( m \) containers, such that \( n > m \), then at least one container contains more than one item.
Counting Hairs

- **Prove.** There exist two people such that both live in London and have exactly the same number of hairs on their head at this specific moment.

Solution

- **Population of London:** about 8.3 million.
- **Average amount of hairs on head (according to random website):**
  - Blondes: 140,000.
  - Black hair: 110,000.
  - Red hair: 90,000.
- **For simplicity, assume that no person has more than 1,000,000 hairs.**
  - By the pigeonhole principle, there must be at least eight people with London, all with the same number of hairs.
An Erdös Initiation Question

• **Prove.** Consider a subset $A \subset \{1,2,3, \ldots, 2n\}$ with $|A| = n + 1$. Then there exist $b, c \in A$ such that $b$ divides $c$.

• **Answer.** Write every $a \in A$ as $2^k m$, where $m$ is odd.
  ◦ There are $n$ possible values for $m$.
  ◦ By the pigeonhole principle, there are two elements of $A$ with the same $m$. One of those divides the other.

Mutual Acquaintances

• **Prove.** Among every six people, it is possible to find either three mutual acquaintances or three non-mutual acquaintances.

• **Solution.** We build a graph.
  ◦ A vertex for every person.
  ◦ An edge between every pair of people who know each other.
Rephrasing the Problem

- **Prove.** In a graph with six vertices, there is either a cycle of length three or three vertices with no edges between them.

- **Solution.** Consider a vertex \( v \).
  - By the pigeonhole principle, there are either at least three vertices adjacent to it, or at least three vertices that are not.

Solution

- Say that \( v \) is connected to three of the other vertices \( u_1, u_2, u_3 \).
  - If no edge exists in the subgraph induced by \( u_1, u_2, u_3 \), then we are done.
  - If there exists an edge between \( u_i \) and \( u_j \), then \( \{v, u_i, u_j\} \) is a cycle of length three.
  - The case where \( v \) is not connected to three vertices is treated symmetrically.
Reminder: Hypercube Graphs

- The $d$-dimensional hypercube graph $Q_d$.
  - Every vertex is a point with $d$ coordinates, each either 0 or 1.
  - Two vertices are adjacent if they have $d - 1$ common coordinates.
  - $2^d$ vertices.
  - $2^{d-1}d$ edges.

Spanning Trees in Hypercubes

- **Theorem.** Let $T$ be a spanning tree of the hypercube graph $Q_d = (V, E)$. Then there exists an edge $e \in E$ such that inserting $e$ to $T$ yields a cycle of length at least $2d$. 
Solution

- We say that two vertices $v, v' \in V$ are opposite if they differ in all $d$ coordinates.
  - For every $v \in V$, we consider the path in $T$ from $v$ to its opposite $v'$. We direct the first edge in this path so that it leaves $v$.
  - A spanning tree has $|V| - 1$ edge, and we just directed $|V|$ of those, so by the pigeonhole principle there is an edge $e \in E$ that was directed twice!

Solution (cont.)

- $(u, v) \in E$ – an edge that was directed twice.
  - It is the first edge in the path in $T$ from $v$ to $v'$ and the first edge in the path in $T$ from $u$ to $u'$.
  - Thus, the path from $v$ to $u'$ and the path from $u$ to $v'$ are disjoint.
  - Each of these paths is of length at least $d - 1$ since $v, v'$ and $u, u'$ are opposite pairs.
  - Adding the edge $(v', u')$ yields a cycle of length at least $2d$ ($(v', u') \in E$ since there is an edge between the two opposite vertices).
An Alternative Definition

- **Pigeonhole principle.** If there are \( n \) pigeons and \( n + 1 \) holes, then at least one pigeon must have at least two holes in it.

Erdős–Szekeres Theorem

- **Theorem.** Every sequence of \( n^2 + 1 \) distinct numbers contains an increasing or decreasing subsequence of length \( n + 1 \).

  - **Example.** 1, 43, 23, 12, 9, 38, 77, 50, 10, 2.
    - Sequence of length 10.
    - Increasing subsequence: 1, 12, 38, 77.
    - Decreasing subsequence: 43, 23, 12, 10, 2.
Solution

- Denote the sequence as \( a_1, a_2, \ldots, a_{n^2+1} \).
  - \( x_i \) – the length of the longest increasing subsequence ending at \( a_i \).
  - \( y_i \) – the length of the longest decreasing subsequence ending at \( a_i \).
- Consider any \( 1 \leq i < j \leq n^2 + 1 \).
  - If \( a_i > a_j \) then \( y_j \geq y_i + 1 \).
  - If \( a_i < a_j \) then \( x_j \geq x_i + 1 \).
  - Either way, we have \( (x_i, y_i) \neq (x_j, y_j) \).

Solution (cont.)

- Denote the sequence as \( a_1, a_2, \ldots, a_{n^2+1} \).
  - \( x_i \) – the length of the longest increasing sequence ending at \( a_i \).
  - \( y_i \) – the length of the longest decreasing sequence ending at \( a_i \).
- For every \( 1 \leq i \leq n^2 + 1 \) we have a unique pair \( (x_i, y_i) \).
  - If there is no sequence of length \( n + 1 \), then by the pigeonhole principle there must be two identical pairs \( (x_i, y_i) = (x_j, y_j) \).
  - But then we get a contradiction!
Homogenous Subsets

- $S$ – a set of “elements” (e.g., vertices).
  - For an integer $1 \leq r \leq |S|$, we let $\binom{S}{r}$ the set of all subsets of $r$ elements of $S$.
  - We give every subset of $\binom{S}{r}$ a color.
  - We say that $S' \subset S$ is $i$-homogeneous if every subset of $\binom{S'}{r}$ is of color $i$.

Example: Subsets of Size 2

- When the subsets are of size two, this is equivalent to coloring edges of a graph.
  - We proved earlier that if $|S| = 6$ and there are two colors, there is a homogeneous subset of size 3.
  - If $|S| = 5$, there might not be such a subset.
Ramsey Numbers

- $r, p_1, ..., p_k$ – positive integers.
- The **Ramsey number** $R(p_1, ..., p_k; r)$ is the minimum integer $N$ such that every coloring of $\binom{\{1, 2, ..., N\}}{r}$ using $k$ colors yields an $i$-homogeneous set of size $p_i$, for some $1 \leq i \leq k$.
  - From the six-people question, we already know that $R(3, 3; 2) = 6$.

Another Example

- The expression $N = R(3, 3, 3, 3; 2)$ means that every coloring of the edges of $K_N$ using four colors contains a monochromatic triangle.
The Case of $r = 2$

- **Claim.**
  \[ R(p_1, p_2; 2) \leq R(p_1 - 1, p_2; 2) + R(p_1, p_2 - 1; 2). \]

- **Proof.** We can think of a graph with edges colored red and blue.
  - Set $N = R(p_1 - 1, p_2; 2) + R(p_1, p_2 - 1; 2)$.
  - We need to show that in any edge coloring of $K_N$:
    - Either there is a red induced subgraph over $p_1$ vertices,
    - Or there is a blue induced subgraph over $p_2$ vertices.

\[ R(p_1, p_2; 2) \leq R(p_1 - 1, p_2; 2) + R(p_1, p_2 - 1; 2). \]

- Set $s = R(p_1 - 1, p_2; 2)$, $t = R(p_1, p_2 - 1; 2)$.
  - $N = s + t$. Consider a red-blue colored copy of $K_N$. Let $v$ be a vertex in $K_N$.
  - Since there are $s + t - 1$ other vertices in $K_N$, $v$ is either incident to at least $s$ red edges or to at least $t$ blue edges.
  - Assume that $v$ is incident to at least $s$ red edges. Thus, the “red neighbors” of $v$ either contain a blue induced subgraph of size $p_2$ or a red induced subgraph $H$ of size $p_1 - 1$.
  - In the former case, we are done. In the latter, adding $v$ to $H$ yields a red induced subgraph of size $p_1$. 
Illustration

- If \( v \) is incident to at least \( s \) red edges, we consider the other endpoint of these edges.
- If they contain a large red induced subgraph, we add \( v \) to it.

The Case of Larger \( r \)

- The previous claim implies that \( R(p_1, p_2; 2) \) is finite for any positive \( p_1 \) and \( p_2 \).
  - We now generalize to the case of larger \( r \).
- **Theorem.** For any positive integers \( r, p_1, p_2 \), the Ramsey number \( R(p_1, p_2; r) \) is finite.
A Bit of Intuition

• If \( r = 3 \), we color triples of vertices.
  ◦ Can be thought of as coloring the triangular faces of \( K_n \).
  ◦ We are looking for a large subset \( S \) of the vertices, such that each triangle that is spanned by three vertices of \( S \) has the same color.

What Do We Need to Prove?

• For any positive integers \( r, p_1, p_2 \), we need to prove that there exists \( N \) such that every coloring of the \( r \)-tuples of a set \( S \) of size \( N \) in red and blue contains:
  ◦ Either a subset of \( p_1 \) elements such that every \( r \)-tuple in it is red,
  ◦ or a subset of \( p_2 \) elements such that every \( r \)-tuple in it is blue.
Proof

- We use a *double induction*.
  - We prove the theorem by induction on $r$.
  - We prove the induction step using an induction on $p_1 + p_2$.

- **Induction basis (on $r$).**
  - When $r = 1$, we can simply take $p_1 + p_2 - 1$ elements.

Induction Step

- We prove the induction step for a given value of $r$ by induction on $p_1 + p_2$.
  - **Induction basis.** If $p_1 < r$ or $p_2 < r$ then the claim vacuously holds for sets of $p_i$ elements and no $r$-tuples.
  - **Induction step.** We set $q_1 = R(p_1 - 1, p_2; r)$, $q_2 = R(p_1, p_2 - 1; r)$, and $N = 1 + R(q_1, q_2; r - 1)$.
  - By the hypotheses, $q_1, q_2$, and $N$ are finite.
Proof (cont.)

\[ q_1 = R(p_1 - 1, p_2; r), \quad q_2 = R(p_1, p_2 - 1), \quad N = 1 + R(q_1, q_2, r - 1). \]

- Let \( S \) be a set of \( N \) elements, with every \( r \)-tuple colored either red or blue.
  - Pick an element \( x \in S \) and let \( S' = S \setminus \{x\} \).
  - We color an \((r - 1)\)-tuple \( T \subset S' \) using the same color as the \( r \)-tuple \( T \cup \{x\} \).
  - Since \(|S'| = N - 1 = R(q_1, q_2, r - 1)\), there is either a red subset of \( q_1 \) elements (with respect to the colors of the \((r - 1)\)-tuples), or a blue subset of \( q_2 \) elements.

Completing the Proof

\[ q_1 = R(p_1 - 1, p_2; r), \quad q_2 = R(p_1, p_2 - 1), \quad N = 1 + R(q_1, q_2, r - 1). \]

- \( S - \) a set of \( N \) elements. \( S' = S \setminus \{x\} \).
- WLOG, we assume that there is a red subset \( S_r \) of \( q_1 \) elements (with respect to \((r - 1)\)-tuples).
- We consider the \( r \)-tuples of \( S_r \). Since \(|S_r| = q_1 = R(p_1 - 1, p_2; r)\), either there is a blue subset of size \( p_2 \), or a red subset of size \( p_1 - 1 \).
- In the latter case, by adding \( x \) to the subset, we obtain a red subset of size \( p \).
Ramsey’s Theorem

- A straightforward extension of the proof yields a more general result.

**Theorem.** For any positive integers \( r, p_1, \ldots, p_k \), the Ramsey number \( R(p_1, \ldots, p_k; r) \) is finite.
  - One way to think of the theorem: in every sufficiently large arbitrary object (i.e., an arbitrary coloring) there must be some order (i.e., a monochromatic subset).

The End

- In the 1950’s, the Hungarian sociologist Sandor Szalai studied friendship relationships between children.
- He observed that in any group of around 20 children he was able to find four children who were mutual friends, or four children such that no two of them were friends.
  - Is this an interesting sociological phenomena?