Reminder: Spanning Trees

- A **spanning tree** is a tree that contains all of the vertices of the graph.
- A graph can contain many distinct spanning trees.
Counting Spanning Trees

- In this class we focus on counting how many spanning trees a graph has.
  - We begin with graphs that we know well:

\[ C_n \quad \text{and} \quad K_n \]

A First Estimate

- **Claim.** \( K_n \) contains at least \( (n - 1)! \) spanning trees.
  - Denote the vertices of \( K_n \) as \( \{v_1, v_2, \ldots, v_n\} \).
  - We start with the edge \( (v_1, v_2) \).
  - For \( 3 \leq i \leq n \), we connect \( v_i \) to a vertex of \( \{v_1, \ldots, v_{i-1}\} \).
Completing the Solution

- Denote the vertices of $K_n$ as $\{v_1, v_2, ..., v_n\}$.
- We start with the edge $(v_1, v_2)$.
- For $3 \leq i \leq n$, we connect $v_i$ to a vertex of $\{v_1, ..., v_{i-1}\}$.
- There are $(n - 1)!$ Distinct spanning trees that can be obtained in this way.

Cayley’s Formula

- **Theorem.** $K_n$ contains exactly $n^{n-2}$ spanning trees.
  - There are $n^{n-2}$ sequences of length $n - 2$ where each entry is a number of $\{1, 2, ..., n\}$.
  - To prove the theorem, we describe a bijection between the set of these sequences and the spanning trees of $K_n$. 

![Diagram of vertex connections](image)
Prüfer Code

- $v_1, v_2, \ldots, v_n$ – the vertices of $K_n$.
- $T$ – a spanning tree in $K_n$.
- Repeat the following until two vertices remain.
  - Remove the leaf $v_i$ with the smallest index.
  - Write down the remaining neighbor of $v_i$ in the tree.

Examples for Small Values of $n$
From a Map to a Bijection

- We have a map from the spanning trees of $K_n$ to the set of sequences.
- To show that this map is a bijection, we show that it has an inverse.
  - We show that given a sequence, we can recover the spanning tree leading to it.
  - We begin with a graph with $n$ isolated vertices, and add edges according to the sequence.

Recovering the First Edge

- Say that the first number in our sequence is 5. Then the first edge that was removed from the tree had $v_5$ as an endpoint.
  - How can we find the other endpoint?
  - It is the leaf with the smallest index. But how can we tell whether $v_1$ is a leaf of the tree?
  - $v_1$ is a leaf if and only if it does not appear in the sequence.
Examples

\[ 7, 4, 4, 7, 5 \], so 1 is a leaf

\[ 1, 2, 1, 3, 3, 5 \], so 1 is not a leaf

Example

- We have 7 vertices and the code \( 7, 4, 4, 7, 5 \).
  - The first edge is connected to \( v_7 \). Since 1 does not appear in the sequence, we know that \( v_1 \) is a leaf of the tree, so this edge is \((v_1, v_7)\).
Recovering the Entire Tree

- We start with a graph with no edges, where every vertex is unmarked, and repeat:
  - Consider the next number $a_i$ in the sequence and connect vertex $v_{a_i}$ to vertex $v_j$ with the smallest index, out of the unmarked vertices that do not appear in the remaining part of the sequence.
  - We then mark $v_j$.
  - $v_j$ must be a leaf in the tree that is obtained by removing the edges that were previously discovered.

Example (cont.)

- We have 7 vertices and the code $7, 4, 4, 7, 5$. 
Example (cont.)

We have 7 vertices and the code \(7, 4, 4, 7, 5\).

\[\begin{array}{cccc}
 v_7 & v_1 & v_6 & v_5 \\
 v_4 & v_3 & v_2 & v_6 \\
 v_5 & v_4 & v_3 & v_2 \\
 v_4 & v_3 & v_2 & v_6 \\
 v_5 & v_4 & v_3 & v_2 \\
\end{array}\]
The Final Step

- The sequence contains only \( n - 2 \) numbers, so there is still one edge missing in the tree.
- The final edge must be between the two remaining unmarked vertices.
- \( 7, 4, 4, 7, 5 \):

Concluding the Proof

- We described a map from the set of spanning trees of \( K_n \) to the set of sequences.
- We proved that this map is a bijection, since it has an inverse.
- Thus, the number of spanning trees of \( K_n \) equals the number of sequences, which is \( n^{n-2} \).
Which Famous Serial Killer is Related to Mathematics

Charles Manson  The Unabomber  Ted Bundy

Spanning Trees in a General Graph

- Given a graph $G = (V, E)$, we denote by $\tau(G)$ the number spanning trees of $G$. 
Recall: Adjacency Matrix

- Consider a graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, \ldots, v_n\}$.
  - The adjacency matrix of $G$ is a symmetric $n \times n$ matrix $A$. The cell $A_{ij}$ contains the number of edges between $v_i$ and $v_j$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 \end{pmatrix}$$

Matrix Tree Theorem

- **Theorem.** Let $G = (V, E)$ be a loopless graph, such that $V = \{v_1, \ldots, v_n\}$.
  - Let $A$ be the adjacency matrix of $G$.
  - Let $D$ be the diagonal matrix with $D_{ii} = \text{deg} v_i$.
  - Let $M = D - A$.
  - For any $1 \leq j \leq n$, removing the $j$’th row and column from $M$ and taking the determinant of the resulting matrix gives $\tau(G)$. 
Example

\[ D = \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix} \quad A = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix} \]

\[ M = D - A = \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 2 & 0 \\
-1 & -1 & 0 & 2
\end{pmatrix} \]

\[ \begin{vmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 2
\end{vmatrix} = 8 \]

Reminder: Incidence Matrix of a Directed Graph

- Consider a directed graph \( G = (V, E) \).
  - We order the vertices as \( V = \{v_1, v_2, ..., v_n\} \) and the edges as \( E = \{e_1, e_2, ..., e_m\} \).
  - The incidence matrix of \( G \) is an \( n \times m \) matrix \( M \). The cell \( M_{ij} \) contains -1 if \( e_j \) is entering \( v_i \), and 1 if \( e_j \) is leaving \( v_i \).
Using the Incidence Matrix

- We arbitrarily direct every edge of $G$. Let $I$ denote the incidence matrix of the resulting graph.
  - Then $M = I \cdot I^T$.

$$I = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Removing a Row and a Column

- After directing $G$, we have $M = I \cdot I^T$.
  - Let $M_j$ denote the matrix obtained by removing the $j$'th row and column from $M$.
  - Let $I_j$ denote the matrix obtained by removing the $j$'th row from $I$. Then $M_j = I_j I_j^T$.

$$I = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$
The Cauchy–Binet Formula

For \( m \geq n \), let \( A \) be an \( n \times m \) matrix and let \( B \) be an \( m \times n \) matrix.

- Let \( S_{m,n} \) denote the set of \( n \)-element subsets of \( \{1,2, ..., m\} \).
- For \( s \in S_{m,n} \), let \( A_s \) denote the \( n \times n \) submatrix containing the columns with indices in \( s \).
- Similarly, let \( B_s \) denote the \( n \times n \) submatrix, containing the rows with indices in \( s \).

**Theorem.**

\[
\det(AB) = \sum_{s \in S_{m,n}} \det(A_s) \det(B_s).
\]

**Claim.** Let \( N \) be an \( (n-1) \times (n-1) \) submatrix of the (directed) incidence matrix \( I \). If the \( n-1 \) chosen columns form a spanning tree then \( \det N = \pm 1 \).

\[
I = \begin{pmatrix}
-1 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1
\end{pmatrix}
\]

\[
N = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 1 \\
1 & -1 & 0
\end{pmatrix}
\]

\[\det N = -1\]
Proof

• **Proof.** By induction on $|V|$.
  
  ◦ **Induction basis.** If $|V| = 2$, then a spanning tree is a single edge. Since this edge is connected to both vertices, every relevant $1 \times 1$ matrix has determinant $\pm 1$.

  $$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

• **Induction step.** Any tree has at least two leaves. Since we removed one row from $I$, there is a row in $N$ corresponding to a leaf $v$.

  • The row of $v$ contains a single non-zero element in column $\ell$.

  $$I = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \quad N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

  $$\det N = -1$$


Completing the Induction Step

• Let $N'$ be the matrix obtained by removing from $N$ the row of the leaf $v$ and the $l'$th column.
  ◦ We have $\det N = \pm 1 \cdot \det N'$.
  ◦ **By the induction hypothesis**, $\det N' = \pm 1$ since $N'$ is an $(n - 2) \times (n - 2)$ submatrix of the incidence matrix of $G - v$.
  ◦ Thus, we have $\det N = \pm 1$.

**Claim.** Let $N$ be an $(n - 1) \times (n - 1)$ submatrix of the (directed) incidence matrix $I$. If the $n - 1$ chosen columns do **not** form a spanning tree then $\det N = 0$.

$$I = \begin{pmatrix}
-1 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1
\end{pmatrix}, \quad N = \begin{pmatrix}
-1 & 0 & 1 \\
0 & 1 & -1 \\
1 & -1 & 0
\end{pmatrix}$$
Proof

- A graph $G = (V, E)$ with $|E| = |V| - 1$ that is not a spanning tree must contain a cycle $C$.
  - We take a linear combination of the column vectors of $N$. If the corresponding edge is not in $C$, its coefficient is 0. If it goes clockwise in $C$, a coefficient of 1. Otherwise, -1.
  - Every row has one 1 and one -1, so the linear combination is zero.
  - Since the columns are dependent, the determinant equals 0.

Concluding the Proof

- We have $M_j = I_j I_j^t$.
- By the Cauchy–Binet Formula
  $$\det M_j = \det I_j I_j^t = \sum_{s \in S_{m,n}} \det \left( (I_j)_s \right) \det \left( (I_j^t)_s \right).$$
- A subset $s \in S_{m,n}$ contributes 1 to the sum iff it corresponds to a spanning tree.
  - Thus, the determinant is $\tau(G)$. 
The End

- Before Ted Kaczynski was the Unabomber, he was a mathematics professor at Berkley.
  - At the time, he was the youngest professor ever to get hired by Berkley.
  - His specialty was complex analysis.