Ma/CS 6b
Class 2: Matchings

By Adam Sheffer

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There aren’t enough crocodiles in the presentations

Only today! 99% off for pirate garden gnomes!

Adam make me a public key!
National Resident Matching Program

- Every medical student who is about to graduate ranks hospitals in which she wants to do her residency.
- Every hospital ranks students that it is interested in.
- Every year, over 20,000 applicants apply to about 1,800 programs.
- How can we handle this?

Bipartite Graphs

- A graph $G = (V, E)$ is bipartite if we can partition $V$ into disjoint subsets $V_1, V_2 \subseteq V$ such that every edge of $E$ is between a vertex of $V_1$ and $V_2$.
- Equivalently, the vertices of $V$ can be colored red and blue such that no edge is monochromatic.
A Useful Graph Family

- $K_{m,n}$ – a complete bipartite graph with $m$ vertices on one side and $m$ on the other (there is an edge between every two vertices on opposite sides).

Reminder: Matchings

- A matching in an graph is a set of vertex-disjoint edges.
- The size of a matching is the number of edges in it.
- A maximum matching of $G$ is a matching of maximum size.
Reminder: Perfect Matchings

• A *perfect matching* of a graph $G = (V, E)$ is a matching of size $|V|/2$.

Back to the Medical Students

• How can we approach our medical students problem?
  ◦ *Bipartite graph* – a vertex in $V_1$ for each student. A vertex in $V_2$ for each hospital.
  ◦ An edge exists between a student and a hospital if they are interested in each other.
Solving the Problem?

- What should we do with the student-hospital graph?
  - We can find the maximum matching, but there are two problems.

First Problem

- **Problem.** Some hospitals might wish to hire more than one resident.
- **Solution.** (as we saw in 6a)
  - If a hospital wants to hire $k$ residents, in the graph we have $k$ vertices for it.
Second Problem

• Problem. We did not consider the rankings of the students and hospitals.
  ◦ We might have chosen the red matching.
  ◦ However, perhaps student $A$ prefers hospital $\beta$, student $B$ prefers hospital $\alpha$, and similarly for the hospitals.

Alternating Paths

• Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
• A path is alternating for $M$ if it starts with an unmatched vertex of $V_1$ and every other edge of it is in $M$.

* Notice that this is definition is different than the one from 6a.
Augmenting Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is augmenting for $M$ if it is an alternating path of $M$, and it ends in an unmatched vertex.

Using Augmenting Paths

- Consider a matching $M$ and an augmenting path $P$ of $M$.
- By switching in $P$ the edges that are in $M$ with the edges that are not, we obtain a larger matching.
Augmenting Paths and Matchings

- **Claim.** Let \( G = (V_1 \cup V_2, E) \) be a bipartite graph and let \( M \) be a matching in \( G \). Then \( M \) is a not a maximum matching iff there exists an augmenting paths for it.

- **Proof.**
  - If there is an augmenting path, we can use it to find a larger matching, so \( M \) is not a maximum matching.
  - It remains to prove that if \( M \) is not a maximum matching, there is an augmenting path for it.

Completing the Proof

- Let \( M^* \) be a maximum matching of \( G \).
- Let \( F \) be the set of edges that are either in \( M \) or in \( M^* \), but not in both. Set \( G' = (V, F) \).
- In \( G' \), every vertex is of degree at most two.
- Thus, \( G' \) is composed of paths, cycles, and isolated vertices. Since \( |M| < |M^*| \), there must be at least one augmenting path for \( M \).
Traffic Cameras

- **Problem.** The city of Pasadena wants to have traffic cameras that cover all of the roads of the city.
  - A camera covers 360° and sees far enough to cover a road at least until the next intersection.
  - How can we efficiently find the minimum number of cameras that are necessary?

Considering the Problem as a Graph

- We build a graph:
  - A vertex for every intersection.
  - An edge between every two adjacent intersections.

- **What do we need to find in the graph?**
  - A minimum set of vertices $S$ such that every edge is adjacent to at least one vertex of $S$. 
Vertex Covers

- Let $G = (V, E)$ be a graph. A vertex cover of $G$ is a set of vertices $V' \subseteq V$ such that every edge of $E$ is incident to at least one vertex of $V'$.

More About Vertex Covers

- No polynomial-time algorithm is known for finding the minimum vertex cover.
- This is a main open problem in theoretical computer science.
  - Significantly easier in bipartite graphs.
König’s Theorem

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then the size of a maximum matching of $G$ is equal to the size of a minimum vertex cover of $G$.

- **Proof.**
  - $m$ - a maximum matching.
  - $v$ – a minimum vertex cover.
  - Since the edges of $m$ are vertex-disjoint and $v$ must contain a vertex of each, we have $|v| \geq |m|$.

Proof (cont.)

- $v$ – a minimum vertex cover.
- $m$ – a maximum matching.
- We saw that $v$ is larger or equal than $m$.
- To complete the proof, it suffices to find a vertex cover of size $|m|$.
- We build a subset $V' \subseteq V$ by taking a vertex out of each edge $e = (a, b)$ of $m$.
  - If an alternating path ends in $b \in m$ we add $b$ to $V'$.
  - Otherwise, we add $a$ to $V'$. 
Proof (cont.)

- $V'$ consists of one vertex of each edge $(a, b) \in m$.
  - If an alternating path ends in $b \in m$ we add $b$ to $V'$.
  - Otherwise, we add $a$ to $V'$.

- Assume, for contradiction, that there is an edge $(a, b) \in E$ that is not covered by $V'$.
  - Either $a$ or $b$ must be matched in $m$, since otherwise $m$ is not a maximum matching.

The Case where $b$ is Matched

- Assume that $b$ is matched in $m$.
  - Then $(a', b) \in m$ for some $a' \in V_1$.
  - Since $b \notin V'$, then $a' \in V'$, and no alternating path ends at $b$.
  - But $(a, b)$ is such an alternating path! Contradiction!
The Case where $a$ is Matched

- Assume that $a$ is matched in $m$.
  - Then $(a, b') \in m$ for some $b' \in V_2$.
  - Since $a \notin V'$, then $b' \in V'$, and there is an alternating path $P$ ending at $b'$.
  - If $(a, b') \notin P$, then the path $P + (a, b') + (a, b)$ is an alternating path ending in $b$. Since $b$ is unmatched, this an augmenting path for $m$, contradicting the maximality of $m$.
  - It cannot be that $(a, b') \in P$. No alternating path can end in $(a, b')$.

Illustration #1

- If $(a, b') \notin P$, then the path $P + (a, b') + (a, b)$ is an alternating path ending in $b$.
Illustration #2

- It cannot be that \((a, b') \in P\). No alternating path can end in \((a, b')\).
  - In the path, we move from \(V_1\) to \(V_2\) only with unmatched edges.

Concluding the Proof

- \(m\) - a maximum matching.
- We defined a subset \(V' \subset V\) of size \(|m|\) and proved that it is a vertex cover.
- We also proved that any vertex cover is of size at least \(|m|\), implying that \(V'\) is a minimum vertex cover.
  - That is, the minimum vertex cover has the same size as the maximum matching.
Vertex Covers in Bipartite Graphs

**Problem.** Describe a polynomial-time algorithm for finding a vertex cover in a bipartite graph $G = (V_1 \cup V_2, E)$.

**Solution.**
- From 6a, we know an algorithm for finding a maximum matching $M$ in a bipartite graph.
- We pick one vertex out of each edge $(a, b) \in M$. If an alternating path ends in $b$ we pick $b$. Otherwise, we pick $a$.
- But how do we know whether such a path exists?

Finding an Alternating Path

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a maximum matching.
- We wish to find whether there is an alternating path for $M$ ending at a vertex $b \in V_2$.
  - We run a variant of BFS from $b$. 

BFS Variant

- The root of the BFS tree is $b$.
- At the first level we have vertices that are adjacent to $b$ in $V_1$.

BFS Variant (2)

- For each vertex of level 1, if it is matched in $M$, we connect it to its match.
**BFS Variant (3)**

- For each vertex of level 2, we connect it (by edges not in $M$) to any of its neighbors in $V_1$ that are not in the tree yet.

![BFS Variant (3) Diagram]

**BFS Variant (4)**

- We repeat this process:
  - Vertices of even levels ($q_i$’s) have as their children every new vertex adjacent to them.
  - Vertices of odd levels ($p_i$’s) have only their matching vertex as a child.

![BFS Variant (4) Diagram]
BFS Variant (5)

• How can we tell whether an alternating path for $M$ end at $b$?
  ◦ Every such path corresponds to an unmatched vertex at an odd level of the tree (i.e., a leaf at an odd level).

Summing up

• Given a bipartite graph $G = (V_1 \cup V_2)$, we find a minimum vertex cover $V'$ of $G$ by
  ◦ Finding a perfect matching $M$.
  ◦ From each edge $(a, b) \in M$, we add one vertex to $V'$:
    • If an alternating path ends in $b \in m$ we pick $b$. Otherwise, we pick $a$.
    • To check whether such a path exists, run the BFS variant that we just saw.
The End

There's a certain type of brain that's easily disengaged.

If you show it an interesting problem, it immediately drops everything else to work on it.

This has led me to invent a new story: NERD SNIPING. See that physicist crossing the road?

Hey!

On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?

It's... hypnotic. Interesting. Maybe if you start with ... no, wait. Hmm... you could-

F0000M

I will have no part in this. Crack, you're a s**t at fun! physicists are two things: mathematicians three.