Ma/CS 6b
Class 1: Graph Recap

By Adam Sheffer

Course Details

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• 1:00 Monday, Wednesday, and Friday.

• http://www.math.caltech.edu/~2014-15/2term/ma006b/
Course Structure

• No exam!
• Grade based on weekly homework assignments.
  ◦ Due by noon on Wednesdays.
  ◦ Please read the homework policy on the website!

• Books
  ◦ Introduction to Graph Theory, 2nd edition, Douglas West.
  ◦ Graph Theory, 4th edition, Reinhard Diestel.

Graphs

- Undirected graph
- Directed graph

- In this class, unless stated otherwise, the graph is undirected.
Graph Representation

- We write $G = (V, E)$. That is, the graph $G$ has vertex set $V$ and edge set $E$.

- Example. In the figure:
  - $V = \{a, b, c, d, e\}$.
  - $E = \{(a, b), (a, d), (a, e), (b, c), (b, e)\}$.

Graph Representation (cont.)

- $V = \{a, b, c, d, e\}$.
- $E = \{(a, b), (a, d), (a, e), (b, c), (b, e)\}$.
Simple Graphs

- An edge is a **loop** if both of its endpoints are the same vertex.
- Two edges are **parallel** if they are between the same pair of vertices.
- A graph is **simple** if it contains no loops and no parallel edges.
- Unless stated otherwise, the graph is simple.

A loop

Parallel edges

Degrees

- The **degree** of a vertex is the number of edges that are adjacent to it.
- **Prove.** In any graph, the **sum of the degrees** of the vertices is even.

**Proof.** Every edge contributes 1 to the degree of exactly two vertices. Thus,

\[ \sum_{v \in V} \deg(v) = \sum_{e \in E} 2 = 2|E|. \]
Subgraphs and Average Degree

- Given two graphs $G = (V, E)$ and $G' = (V', E')$. We say that $G'$ is a **subgraph** of $G$ if $V' \subseteq V$ and $E' \subseteq E$.
  - We say that $G'$ is an **induced subgraph** on $V' \subset V$ if $E'$ contains exactly the edges of $E$ that connect two vertices of $V'$.

- The **average degree** of a graph $G = (V, E)$ is
  \[
  \text{deg}(G) = \frac{1}{|V|} \sum_{v \in V} \text{deg}(v) = \frac{2|E|}{|V|}.
  \]

Induced Subgraphs

- Which of these is an induced subgraph?
  - \[\text{Correct} \quad \text{Incorrect} \quad \text{Incorrect} \]
## Subgraphs with Large Degrees

**Problem.** For every graph $G = (V, E)$ with $|E| \geq 1$ there exists an induced subgraph $H$ of $G$, such that the minimum degree in $H$ is larger than $\deg(G)/2$.

![Graph transformation](image)

\[
\frac{|E|}{|V|} = \frac{7}{6}
\]

## Solution

- Set $k = \deg(G)/2 = |E|/|V|$.
- We repeatedly remove vertices of degree at most $k$ until none are left.
  - If no vertex of $G$ has degree $\leq k$, we are done.
  - **Otherwise, how can we make sure that we do not remove the entire graph?**
    - Denote our sequence of subgraphs as $G = G_0, G_1, G_2, ..., G_m$.
    - At each step, we remove one vertex and at most $k$ edges. We thus have $\deg(G_m) \geq \deg(G_{m-1}) \geq \cdots \geq \deg(G_1) \geq 2k$. 
Solution (cont.)

- Denote our sequence of subgraphs as $G = G_0, G_1, G_2, \ldots, G_m$.
- At each step, we remove one vertex and at most $k$ edges. We thus have $\deg(G_m) \geq \deg(G_{m-1}) \geq \cdots \geq \deg(G_1) > 2k$.
- Since $\deg(G_m) \geq 2k$, it must contain edges.

Paths and Cycles:

A **cycle** is a path that starts and ends in the same vertex.

Path between $a$ and $b$. 

Cycle through $a$. 

A **cycle** is a path that starts and ends in the same vertex.
More on Paths and Cycles

- A path/cycle is said to be simple if it does not visit any vertex more than once.

- The length of a path/cycle is the number of edges that it consists of.

- **Example.** A simple cycle of length 5.

Connected Graphs

- A graph $G = (V, E)$ is connected if for any pair $u, v \in V$, there is a path in $G$ between $u$ and $v$. 
Connected Components

• A **connected component** (for short, **component**) of a graph $G = (V, E)$ is a maximal connected subgraph of $G$.
  ◦ For example, a graph with three connected components:

Paths and Degrees

• **Problem.** Let $G = (V, E)$ be a graph such that the degree of every $v \in V$ is at least $d$ (for some $d \geq 2$). Prove that $G$ contains a path of length $d$. 

A graph with minimum degree 3.
Proof

- Assume, for contradiction, that a longest path $P$ is of length $c < d$.
- Consider a vertex $v$ which is an endpoint of $P$.
- Since $\deg v \geq d \geq c + 1$, it must be connected to at least one vertex $u \notin P$.
- By adding the edge $(v, u)$ to $P$, we obtain a longer path, contradicting the maximality of $P$.

A Variant of the Problem

- **Problem.** Let $G = (V, E)$ be a graph such that the degree of every $v \in V$ is at least $d$ (for some $d \geq 2$).
  - Then there exists a cycle of length at least $d + 1$.
Distance

- Consider an undirected graph $G = (V, E)$ and two vertices $v, u \in V$.
- The distance in $G$ between $u$ and $v$, denoted $d(u, v)$ is the length of the shortest path between the two vertices.
- The diameter of $G$ is the maximum distance between two vertices of $G$
  - That is, $\max_{u,v} d(u, v)$.

Diameter and Cycles

- **Prove.** Let $G = (V, E)$ be a graph of diameter $D$ that contains at least one cycle. Then $G$ contains a cycle of length at most $2D + 1$.

- **Example.**
  - Diameter: 2
  - Length of shortest cycle: 5
Proof

• Assume, for contradiction, that the length of the shortest cycle $C$ is at least $2D + 2$.
  ◦ Consider two vertices $u, v$ with a distance of $D + 1$ in $C$.
  ◦ If there is no shorter path between $u$ and $v$, we get a contradiction to the diameter being $D$.
  ◦ If there is a shorter path between $u$ and $v$, we get a contradiction for $C$ being the shortest cycle in $G$.

More Degrees and Distances

• Problem. Consider a graph $G = (V, E)$ and integers $k, d \geq 3$, such that
  ◦ The degree of every vertex of $V$ is at most $d$.
  ◦ There exists a vertex $v \in V$ such that for every $u \in V$ we have $d[u, v] \leq k$.
What is the maximum number of vertices that $V$ can contain?
Solution

• We partition the vertices of $V$ according to their distance from $v$:
  ◦ How many vertices satisfy $d[v, u] = 0$? 1
  ◦ How many vertices satisfy $d[v, u] = 1$?
    • At most $d$.
  ◦ How many vertices satisfy $d[v, u] = 2$?
    • At most $d(d - 1)$.
  ◦ How many vertices satisfy $d[v, u] = i$?
    • At most $d(d - 1)^{i-1}$, for every $1 \leq i \leq k$.

Solution (cont.)

• We have the bound
  \[ |V| \leq 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1} \]
  \[ = 1 + d \frac{(d - 1)^k - 1}{(d - 1) - 1} = \frac{d(d - 1)^k - 2}{d - 2}. \]

• Is this tight?
  ◦ Yes
Trees and Forests

- In a graph, a **tree** is a connected subgraph containing no cycles.
- A **forest** is a set of non-connected trees.

Leaves

- Given a tree $T$, a **leaf** of $T$ is a vertex of degree 1.
- **Claim.** Every tree contains a leaf.
- **Proof.** Consider a vertex $v$ in $T$.
  - If $v$ has degree 1, we are done.
  - Otherwise, we travel the tree without crossing any edge more than once.
  - No vertex is visited twice since there are no cycles in $T$. Thus, eventually we will get stuck.
  - The vertex that we got stuck in is of degree 1.
The Size of a Tree

- Given a tree with \( n \) vertices, how many edges are in it?
  - Exactly \( n - 1 \).
  - Proof sketch. By induction. By removing a leaf we obtain a tree by one vertex and one edge.

Unique Paths in Trees

- Claim. In any tree \( T \) there is exactly one path between any two vertices.
  - Assume, for contradiction, that there are two paths \( P, Q \) in \( T \) between vertices \( s \) and \( t \).
  - \( u \) – the last common vertex before the paths \( P, Q \) split (when traveling from \( s \) to \( t \)).
  - \( v \) – the first vertex common to both paths after \( u \).
  - The portions of \( P \) and \( Q \) between \( u \) and \( v \) form a cycle. Contradiction!
Rooted Trees

- A **rooted tree** is a tree with a special vertex – the **root** – that is singled out.
- We draw the tree with the root on top, and the edges “grow downwards”.
- A vertex \( v \) is the **parent** of a vertex \( u \) if there is an edge \((u, v)\) and \( v \) is above \( u \).
  - Each vertex, except for the root, has a **unique parent**.

\[ s \text{ is the root and } t's \text{ parent} \]

The BFS Algorithm

- The BFS algorithm receives a graph \( G = (V, E) \) and a vertex \( s \in V \).
  - It outputs a **BFS tree**, containing shortest paths from \( s \) to any vertex reachable from \( s \).
  - A **rooted tree with root** \( s \).
Levels of the BFS Tree

- The $i^{th}$ level of the BFS tree is the set of vertices $v \in V$ that satisfy $d(v) = i$.

* This is the origin of the name Breadth First Search.

The End

AND OVER THERE WE HAVE THE LABYRINTH GUARDS. ONE ALWAYS LIES, ONE ALWAYS TELLS THE TRUTH, AND ONE STABS PEOPLE WHO ASK TRICKY QUESTIONS.