Fun with series

or

some sums

For a justification of some of the operations on infinite series of functions used, see Apostol [1, Chapter 11].

1 Geometric series

You already know this series. I am including it for the sake of completeness.

Let $0 < p < 1$.

\[
\sum_{k=0}^{n} p^k = \frac{1-p^{n+1}}{1-p} \tag{1}
\]

\[
\sum_{k=1}^{n} p^k = \frac{p-p^{n+1}}{1-p} \tag{2}
\]

\[
\sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \tag{3}
\]

\[
\sum_{k=1}^{\infty} kp^k = \frac{p}{1-p} \tag{4}
\]

Proof: It is enough to prove (1), so let $x = 1 + p + \cdots + p^n$. Then simply expanding $(1-p)x$ yields $(1-p)x = (1-p)(1+p+\cdots+p^n) = 1 - p^{n+1}$, from which (1) follows. \qed

2 Expected value of a geometric random variable

Let

\[ P\{X = k\} = \frac{1-p}{p} p^k, \quad k = 1, 2, \ldots \]

By (4), these probabilities sum to 1. Thus

\[ EX = \frac{1-p}{p} \sum_{k=1}^{\infty} kp^k. \]
I claim that

\[
\sum_{k=1}^{\infty} kp^k = \frac{p}{(1 - p)^2}. \tag{5}
\]

So

\[
E X = \frac{1 - p}{p} \sum_{k=1}^{\infty} kp^k = \frac{1}{1 - p}.
\]

For example, the expected length of the St. Petersburg game (toss a coin until the first Tails) has \( p = 1/2 \), so the expected length is \( 1/(1 - (1/2)) = 2 \).

The elementary proof of (5): Let \( x = p + 2p^2 + \cdots + np^n \). Then \((1 - p)x\) expands to

\[
(1 - p)x = p + 2p^2 + 3p^3 + \cdots + np^n
- \ p^2 - 2p^3 - \cdots - (n - 1)p^n - np^{n+1} =
\]

\[
= p + p^2 + p^3 + \cdots + p^n - np^{n+1}
= \frac{p - p^{n+1}}{1 - p} - np^{n+1} \quad \text{by (2)}.
\]

Letting \( n \to \infty \) and dividing both sides by \( 1 - p \) gives (5).

Generating function approach to (5): Let \( f(p) = 1/(1 - p) \). By long division, we have (for \( 0 < p < 1 \)) the familiar formula

\[
\frac{1}{1 - p} = f(p) = 1 + p + p^2 + \cdots
\]

Differentiating term-by-term we have

\[
\frac{1}{(1 - p)^2} = f'(p) = 0 + 1 + 2p + 3p^2 + \cdots
\]

\[
= \frac{1}{p} \left( p + 2p^2 + 3p^3 + \cdots \right)
\]

So multiplying both sides by \( p \) gives (5).

3 An inverse expectation

For a geometric \( X \) as above, what is

\[
E \frac{1}{X} = \frac{1 - p}{p} \sum_{k=1}^{\infty} \frac{p^k}{k}?
\]
I claim that

\[ \sum_{k=1}^{\infty} \frac{p^k}{k} = \ln \left( \frac{1}{1-p} \right). \]

(6)

So

\[ E \frac{1}{X} = \frac{1-p}{p} \ln \left( \frac{1}{1-p} \right). \]

Proof of (6) provided by TA Victor Kasatkin: Let

\[ f(p) = \sum_{k=1}^{\infty} \frac{p^k}{k}. \]

It is analytic for \( |p| < 1 \). So for \( |p| < 1 \) we may compute the derivative term-by-term:

\[ f'(p) = \sum_{k=1}^{\infty} p^{k-1} = \sum_{j=0}^{\infty} p^j = \frac{1}{1-p}. \]

Also \( f(0) = 0 \), and thus

\[ f(p) = \int_{0}^{p} f'(t) \, dt = \int_{0}^{p} \frac{1}{1-t} \, dt = -\ln(1-p), \]

so

\[ \sum_{k=1}^{\infty} \frac{p^k}{k} = -\ln(1-p) = \ln \left( \frac{1}{1-p} \right). \]

4 The Taylor series for the exponential

Apostol [1, p. 436] proves that the Taylor series for the exponential function yields the following identity.

For each real number \( x \),

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \]

(7)

Consider the function \( g(x) = e^x \). Its \( n \)th derivative is given by \( g^{(n)}(x) = e^x \), so \( g^{(n)}(0) = 1 \) for every \( n \), and the infinite Taylor’s series expansion of \( g \) around zero is

\[ g(x) = g(0) + \sum_{k=1}^{\infty} \frac{1}{k!} g^{(k)}(0)(x-0)^k = 1 + \sum_{n=1}^{\infty} \frac{x^n}{k!}. \]

So

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \]
References