Ma 145a: Homework set 5, Due December 10 at noon

Choose three from the following five problems to turn in. In this set we discuss some fact regarding k-representations of a finite group G while k has positive characteristic.

1. Find $R_{\mathbb{R}}(\mathbb{Z}_3)$ and $R_{\mathbb{R}}(A_4)$.

2. Let R be a discrete valuation ring (local commutative principal ideal domain which is not a field) with fraction field K of characteristic zero. Assume π_R is a uniformizer (i.e. it generates the unique maximal ideal of R) and $k = R/\pi_R R$ is its residue field. Let G be a finite group. Show that

   (a) any finitely generated $K[G]$-module V contains a lattice which is G-invariant.

   (b) the image of the $k[G]$-module $L/\pi_R L$ in $R_k(G)$ is unique for L any such G-invariant lattice. (This is the same as saying the composition factors are unique up to isomorphism.)

   (A lattice L of V is a free R-submodule of V such that a K-basis of V generates L.)

   Some counterexamples: Take G to be the 2-group $\{1, \sigma\}$ of order 2 and take R to be the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at the prime ideal (2). Then $K = \mathbb{Q}$ and $k = \mathbb{F}_2$. Consider $V = \mathbb{Q}e_1 + \mathbb{Q}e_2$ with $\sigma e_1 = e_2$. Take $L_1 = R(e_1 + e_2) + R(e_1 - e_2)$ and $L_2 = Re_1 + Re_2$. We have $2L_1 \subset L_2 \subset L_1$ but $L_1 = R(e_1 + e_2) \oplus R(e_1 - e_2)$ is decomposable while L_2 is indecomposable but contains the $R[G]$-module $R(e_1 + e_2)$ and is not simple. ($L_1 \simeq L_2$ as $R[G]$-modules) We have $[L_1] = [L_2] = [R(e_1 + e_2)] + [R(e_1 - e_2)]$. In this case, $R_k(G) \simeq \mathbb{Z}$ and for all R-module L of finite length, $[L/2L] \mapsto \dim_k L/2L$.

3. Assume G is a p-group with order p^r and k is an algebraically closed field of characteristic p. Classify all simple $k[G]$-modules.

4. Assume F is an arbitrary field and G is a finite group. Let V be a simple $F[G]$-module. Let N be a normal subgroup of G and W a simple $F[N]$-submodule of V and let $K = Norm_G(W, N)$ be the subgroup $\{g \in G \mid gW \simeq W\}$. Write $\tilde{W}$ for the W-isotypic part of V. Show that $\tilde{W}$ is a simple $F[K]$-module and $V \simeq \oplus_{g \in G/K} g\tilde{W}$, i.e. $V \simeq \mathrm{Ind}_K^G \tilde{W}$.

5. Let R be a commutative ring, and let P be a finitely generated $R[G]$-module which is projective over R. Show that P is a projective $R[G]$-module if and only if for all maximal ideal $\mathfrak{m}$ of R, $P/\mathfrak{m}P$ is a projective $(R/\mathfrak{m})[G]$-module. (A left A-module M is projective if the functor $Hom_A(M, -)$ is exact.)