Ma 145a: Homework set 4, Due November 24 at noon

Choose three from the following four problems to turn in.

1. Recall that the Littlewood-Richardson numbers $N_{\lambda\mu\nu}$ appear in the coefficients of the expansion $S_\lambda S_\mu = \sum_\nu N_{\lambda\mu\nu} S_\nu$ and equal to the multiplicity
$$N_{\lambda\mu\nu} = \dim \mathrm{Hom}_{S_\lambda \times S_\mu}(V_\lambda \boxtimes V_\mu, V_\nu).$$
Show the following identity on Weyl's modules
$$\mathbb{S}_\nu(V \oplus W) = \bigoplus_{\lambda,\mu} N_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W).$$

2. Use the previous result to deduce the Branching law for $GL_n(\mathbb{C})$:
$$\mathrm{Res}^{GL_{n+1}(\mathbb{C})}_{GL_n(\mathbb{C})} \mathbb{S}_\nu(\mathbb{C}^{n+1}) = \bigoplus_{\substack{\lambda \\ \nu_1 \geq \lambda_1 \geq \ldots \geq \nu_n \geq \lambda_n \geq \nu_{n+1}}} \mathbb{S}_\lambda(\mathbb{C}^n)$$
by embedding GL_n into GL_{n+1} via $GL_n \hookrightarrow GL_n \times GL_1 \hookrightarrow GL_{n+1}$.

3. In the lecture we introduced a pairing $\langle\ ,\ \rangle$ on the symmetric polynomials defined by assigning $\langle H_\lambda, M_\mu \rangle = \delta_{\lambda\mu}$. Under this pairing $\langle S_\lambda, S_\mu \rangle = \delta_{\lambda\mu}$ and the Schur polynoimals are dual and give an orthonormal basis. The ring of symmetric polynomials is a graded ring with H_m and E_m of degree m. Consider the graded ring
$$\Lambda = \mathbb{C}[H_1, H_2, ...] = \mathbb{C}[E_1, E_2, ...]$$
and define a ring homomorphism $\partial : \Lambda \to \Lambda$ by
$$\partial(H_m) = E_m.$$
Then a fact is $\partial(S_\lambda) = S_{\lambda'}$ where λ' denotes the conjugate partition of λ. (This is by the identities $S_\lambda = |H_{\lambda_i+j-i}| = |E_{\lambda'_i+j-i}|$ which are known as the Giambelli's formula from geometry.)

   (a) Show that ∂ is an involution $\partial^2 = \partial$.

   (b) Show that $E_{\lambda'} = \sum_\mu K_{\mu'\lambda'} S_\mu$. (This is a dual statement to $H_\lambda = \sum_\mu K_{\mu\lambda} S_\mu$.)

   (c) Show that

   - $Sym^{\lambda_1} V \otimes Sym^{\lambda_2} V \otimes \cdots \otimes Sym^{\lambda_k} V \simeq \bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu V$
   - $\bigwedge^{\lambda'_1} V \otimes \bigwedge^{\lambda'_2} V \otimes \cdots \otimes \bigwedge^{\lambda'_k} V \simeq \bigoplus_\mu K_{\mu'\lambda'} \mathbb{S}_\mu V.$

   (d) Deduce that V_λ is the only irreducible representation of S_n that occurs in both $\mathbb{C}[S_n]a_\lambda$ and $\mathbb{C}[S_n]b_\lambda$ and it occurs with multiplicity 1.

4. The problem is aimed to prove that each $f \in R(G)$ such that $f(1) = 0$ is a $\mathbb{Z}$-linear combination of the element of the form $\mathrm{Ind}_E^G(\alpha - 1)$, where E is an elementary subgroup of G and α is a character of degree 1.

   (a) Let $R'_0(G)$ be a subgroup of $R(G)$ generated by the $\mathrm{Ind}_E^G(\alpha - 1)$'s, and let $R'(G) = \mathbb{Z} + R'_0(G)$. Show that if H is a subgroup of G, Ind_H^G maps $R'_0(H)$ into $R'_0(G)$.

(b) Suppose that H is normal in G and that G/H is abelian. Show that Ind_H^G maps $R'(H)$ into $R'(G)$. (It is enough to show that $\mathrm{Ind}_H^G 1$ belongs to $R'(G)$. Then this follows that $\mathrm{Ind}_H^G 1$ is the sum of $(G : H)$ characters of degree 1 whose kernel contains H.)

(c) Suppose G is elementary (so nilpotent). Let Y be the set of all maximal subgroups of G. Notice that all maximal subgroups of an elementary group are normal of prime index. Deduce that $R(G)$ is generated by characters of degree 1 and $\bigoplus_{H \in Y} \mathrm{Ind}_H^G R(H)$. Then use the previous results to show that $R(G)' = R(G)$ by applying induction on $|G|$.

(d) Use Brauer's Theorem to show that for general finite group G, $R(G)' = R(G)$.