Ma 145a: Homework set 3, Due November 7 at noon

This is a problem set on the Young symmetrizers c_λ. We write G for S_n and a_λ and b_λ are as defined in class. A reference to the content and notations is Fulton and Harris, Representation Theory.

We let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$, be a partition of n. Then the Young subgroup S_λ of S_n is the group $S_{\lambda_1} \times \cdots \times S_{\lambda_k} \hookrightarrow S_n$.

1. Show that as $\mathbb{C}[G]$-modules $\mathbb{C}[G]a_\lambda b_\lambda$ is isomorphic to $\mathbb{C}[G]b_\lambda a_\lambda$.

2. Suppose $\sigma_{\vec{i}} \in S_n$, $\vec{i} = (i_1, i_2, ..., i_n)$ with i_j nonnegative integers such that $\sum_{j=1}^n j i_j = n$, has cycle decomposition product of i_1 many 1-cycles, i_2 many 2-cycles,..., and i_n many n-cycles. Show that the character of $U_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} 1$ at $\sigma_{\vec{i}}$ is

$$\chi_{U_\lambda}(\sigma_{\vec{i}}) = \text{coefficient of } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \text{ in the polynomial } \prod_{j=1}^n (x_1^j + \cdots + x_k^j)^{i_j}.$$

3. Show that $U'_{\lambda'} = \mathbb{C}[G]b_{\lambda'}$ is the induced representation $\mathrm{Ind}_{S_\lambda}^{S_n} sign \otimes sign \otimes \cdots \otimes sign$ by the tensor product of the sign representations for each S_{λ_i}. Here λ' is the partition such that the Young diagram $[\lambda']$ is the conjugate of the Young diagram $[\lambda]$.

4. Show that the complex number n_λ such that $c_\lambda^2 = n_\lambda c_\lambda$ is $n!/\dim V_\lambda$ where $V_\lambda = \mathbb{C}[G]c_\lambda$.