Ma 145a: Homework set 2, Due October 20 at noon

All representations are complex representations. Let $\mathbb{F}$ denote a finite field of order q and $G = GL_2(\mathbb{F})$.

1. Let M be the group consists of elements $\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}) \mid a \in \mathbb{F}^\times, b \in \mathbb{F}\right\}$ in G. Then M contains a normal subgroup $N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}\right\} \cong \mathbb{F}$. Describe all irreducible representations of M.

2. Let $B = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}) \mid a, d \in \mathbb{F}^\times, b \in \mathbb{F}\right\}$ be a subgroup of G.

   (a) Deduce that $GL_2(\mathbb{F})$ has $(q^2-1)(q^2-q)$ elements with q^2-1 conjugacy classes.

   (b) Let $\chi : B \to \mathbb{C}^\times$ be a representation of B of degree 1 defined by

   $$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$$

   for some group homomorphisms $\chi_i : \mathbb{F}^\times \to \mathbb{C}^\times$ for $i = 1, 2$. Show that $\mathrm{Ind}_B^G \chi$ is reducible if and only if $\chi_1 = \chi_2$. If π is an irreducible representation such that $\pi^N \neq 0$ then it occurs in $\mathrm{Ind}_B^G \chi$ for some χ. Such representations are called *principle series*.

   (c) Show that every irreducible representation ρ with $\rho^N \neq 0$ has $\mathrm{Hom}_G(\rho, \mathrm{Ind}_B^G \chi) \neq 0$ for some representation χ of B described in (b).

   (d) The representation $\mathrm{Ind}_B^G 1_B$ can be decomposed as

   $$\mathrm{Ind}_B^G 1_B = 1_G \oplus \mathrm{St}_G$$

   for a unique representation St_G, called the Steinberg representation. Find the character of St_G and show it is irreducible.

   (e) Assume $q \geq 3$. Show that if $\rho : G \to \mathbb{C}^\times$ is a representation of degree 1, then ρ factors through the determinant map $\det : G \to \mathbb{F}^\times$ and equals to $\chi_0 \circ \det$ for some homomorphism $\chi_0 : \mathbb{F}^\times \to \mathbb{C}^\times$.

   In this problem, up to isomorphism, we obtain $(q^2+q-2)/2$ irreducible representations of G. The rest of the irreducible representations of G are said to be *cuspidal*.